Everymind's Pre-Euclid

Imaginative Geometry

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Dedication

To **William George Spencer**

and his book, *Inventional Geometry*,

from which I have borrowed unapologetically,

and to **Isaac Todhunter**,

who taught me to love Euclid.

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Instructions

For Learners

This book is an introduction to the ideas of Euclid's first and most idea-filled book. There are many explanations of his ideas in this text and even more numerous questions, most of which are exercises in imagination. It contains no answers. In a classroom, the teacher would have the understanding (one would hope) to know the answers. But the answers follow, with absolute geometric logic, from the explanations.

And that kind of logic is merely the way we all think when we make sure that we can demonstrate the truth of what we have in our heads. Mostly, our thinking is way sloppier than that. But we can all demonstrate the truth we discover, if we choose not to settle for less.

So you can study this book on your own and should find it an enjoyable introduction to Euclid. There are no proofs and you will not be required to make any proofs either. This text is here to help you understand the ideas which are already proven in Euclid. And to the extent that you understand the simple explanations, you will be able to imagine and express the correct solutions.

In mathematics, we speak of mathematical maturity. This is mostly the ability to be honest with yourself and others. When you produce a solution to a problem, if you are mathematically mature, you know whether your solution is correct or not. If it is not correct, you know pretty accurately how correct it is. Sometimes it happens that you learn you were mistaken in your belief that a solution was correct. This is natural and unavoidable and is the only way to learn.

But you can be mathematically mature **now**. Simply be honest with yourself. If you do this, your "grades" will take care of themselves. Grades are stupid. Just learn to accept that you only know what you can correctly demonstrate. And then demonstrate all you can.

For Teachers

My original idea was to take William Spencer's approach, in his *Inventive Geometry*, to teaching the ideas of Euclid's first six books and to re-focus it in such a way that younger students would discover geometric thinking. His book focused on utility; this book focuses on preparation for pure geometry. The result is a text that introduces the ideas of Euclid's Book I in a way that should be suitable for anyone, of any age, who is ignorant of pure geometry and who would like to know Euclid better.

And Book I does contain his major ideas. It is his triangle book and Euclid proves almost everything with triangles, even when using a previous result without triangles would be far shorter and easier. I make slight in-roads on Books II, III, IV, VI, and even XI -- but without going any further than Book I will easily take you. Basically, we're doing Book I here.

If a school did not require a metric of grades for every course, this would be a course which didn't need grades. The goal is for the students to learn to express the major ideas of Euclid, using their imaginations, so that these ideas are in their minds, ready to be enlarged upon in a rigorous way.

I'm sure you will have to grade your students. Grade them gently so as not to spoil their fun. There is no reason why the sections marked "Quiz-Time" cannot be an enjoyable group effort. For younger students, they should probably be just that. You and this book should teach everyone that geometry is fun.

And the fun lies in understanding an idea so clearly that one can usefully express that idea with originality to obtain a concrete result. The fun lies in realizing that you have dominion over these ideas in your mind and can use your imagination to create all kinds of results using these ideas as tools.

Finally, no answers are provided for the exercises and no teacher's edition will be created. Do what you can. Do what you actually understand. Or, do something you don't understand, admitting your ignorance to your students, and see if you can't all solve it together. If you can't solve it together, then figure out how close to a solution you are. That would be real mathematics. And your goal is to create real mathematicians, right?

Oops, one last thing: **do not over-help your students**. There is no virtue in finishing the book if no one got to think for themselves in the process.

Euclid's Rules

For this book you will need your imagination and a few tools to express your imagination with: a ruler, a compass, and a protractor. I use a six-inch clear ruler marked in inches and centimeters. Clear rulers are more useful than colored or solid ones. I have a little three-inch clear protractor. And I use a nice little compass that holds a pencil stub. But you can use any kind of ruler, compass, and protractor that you can get your hands on, the smaller the better.

Over two thousand years ago, Euclid put almost all the geometry in the world into twelve books which as a whole are called *Euclid's Elements*. We are going to express the ideas from Euclid's Book I using this book and your imagination. But first we need to know the rules of Euclid's pure geometry.

1. So, first, can you imagine two circles that overlap? Where do they overlap?

Let us check your imagination.

2. Can you take your compass and draw those circles you imagined? Do they look as they did in your imagination? If not, what did you miss?

When circles overlap, they **intersect** or **cut** one another at two points. **Points**, in Euclid, are locations. They answer the question, "Where?" So in Euclid's and in our own imagination, points have no dimensions -- no length, width, or depth. They don't add anything to the picture. They only say "here."

3. Can you make points on your paper where your compass was stuck in the paper for your circles?

Oops. Your points have length and width, like tiny, blobby circles. But that's okay. Euclid's ideas are **ideals**. We know exactly what an ideal means. But we can't quite make a perfect example of one in our work. So we do the best we can.

Points have names, or labels, from the alphabet.

4. Can you make points at the intersections of your circles and label them A and B? Can you label the centers P and Q?

Cool. Now we can use our geometric notation to say what we just did. Notation is a lazy way to say something **exactly** so that there can be **no misunderstanding**.

In English, we could say what we did like this:

"The circle with center P has intersected the circumference of the circle with center Q at points A and B and the circle with center Q has also intersected the circumference of the circle with center P at those same points."

Or, using notation, we can be lazy and say it like this:

⊙P × ⊙Q @ A,B

This says exactly the same thing as the English version and, if you know the notation, it cannot be misunderstood. It says, precisely: "Circle, center P, intersects circle, center Q, at A and B." We don't have to choose which words to use and then maybe choose a bad one and possibly confuse someone. If you know the notation, you know exactly which circles intersect and where.

And, already, you know this much notation. You know that "⊙" is "circle," "×" is "intersects," and "@" is "at."

5. Can you make a point A, a point B anywhere else but A, and a line, small enough for your compass to reach both ends, with labels C and D at its ends?

Here's what you just did: ∀A,B ∀CD

You chose and drew any A and any other B and any line CD. Now C and D are points. When you connect them, you have any line CD, with C and D on its ends. When we want to connect two points, like C and D, we say, "Join CD." So we could also say that you did this: ∀A,B,C,D Join CD.

The "∀" is notation for "any", "every", and "all." In pure geometry, these mean the same thing. If something is true for **any** line CD, it is true for **every** line CD -- so it is true for **all** lines, because C and D are just labels.

We will use your A, B, and CD to make some circles. But first I will tell you Euclid's Unbreakable Law:

You cannot use your ruler or compass to measure anything.

Now use your compass to measure CD by putting the point on C and the pencil on D. Oops. We broke the law! But that's okay. The law only applies when you are proving something. And everything in this book was proven 2400 years ago. So we're cool.

6. Can you make a circle on A and a circle on B, each with radius CD? A **radius** is any line from the center of a circle to its circumference.

You just did this: ∀⊙A,CD ∀⊙B,CD -- right?

So, Quiz-Time!

7. If a radius is any line from center to circle, can one circle have any different lines for this or must they all be the same?

8. Can you do this with the same CD:

⊙P,CD × ⊙Q,CD @ A,B
Join AB

Euclid has three rules which we call "postulates." In this book we call them "rules" so we don't misspell "postulates." For the circles, you used Rule No. 3:

Any point and any line can be used to make a circle.

For line AB you used Rule No. 1:

Between any two points, one, and only one, line may be drawn.

Whoa, whoa, whoa. Between A and B we definitely have three lines.

9. Can you say which line on A and B is a Euclidean line? What are the other two called?

Technically, the other two are "arcs." An **arc** is any part of a circle. In Euclid, all lines are drawn with a ruler and all curves are drawn with a compass. So all lines are straight and all curves are arcs of a circle. Most geometry books make a big deal about calling every line a "straight line" which just makes it harder to take notes.

10. So why do grown-ups make things more complicated than they need to? Try to grow up without being this way.

11. Is every Euclidean line a straight line? And is this a stupid question?

Of course it is. Just remember, you may have to say "straight line" in public.

Okay, we still need Rule No. 2:

Any line can be made indefinitely longer in either direction.

We have our line AB. If we want it to start at A and go on past B to some C, we say:

AB(pr) to C

In English, that means, "Line AB produced to C" or "Produce AB to C." If we want to go the other way to some D:

BA(pr) to D

If we wanted to just make it longer in both directions:

AB(pr2)

12. Can you do AB(pr2) on your AB in the circles until you run out of paper? Do not mark on your desk, your clothing, or each other.

13. Is AB(pr2) still straight? If not, are you not embarrassed?

Some people think that "indefinitely" in Rule No. 2 means "infinitely." The word "infinitely" means that it goes on forever.

14. Can you do this: $\forall$ AB, produce AB infinitely?

Surprise! You ran out of paper. And out of desk. And out of other kids to draw on and ran into the wall.

I hope no one got hurt. You cannot draw an infinite line or count your way to infinity or count your way out of infinity, if that's where you are one morning when you wake up.

But you **can** be **drawing** an infinite line or **counting** your way to infinity. And that's all any mathematician can do.

Let's imagine that you are drawing an infinite line. Your little brother or sister or some other little kid you can't stand is following you and counting the inches in your line. You are really amazing. You draw and draw. The annoying brat behind you is counting and counting. And driving you crazy. But you can't stop to run the brat off because then your infinite line would end.

So you draw and draw. Empires rise and fall. Ice Ages and their glaciers come and go. Oh, no! One day the sun goes out! The Earth turns into a ball of ice. Both of you are popsicles. And the infinite drawing and the infinite counting come to an end.

What can you do? At least you won't have to listen to that gazillion-year-old annoying brat count any more.

There is a point to this story and the point belongs to Ludwig Wittgenstein. (A "w" in German sounds like "v".) He said:

Infinity is an adverb.

Wittgenstein is correct here. But if you say this to an adult and they say you are wrong, here's what you do:

1. **Look at the ground.**
2. **Put a thoughtful look on your face.**
3. **Nicely and slowly, say, "Perhaps you should read more Wittgenstein."**

If you do this right, you will get a reputation for being very smart, which is useful. If you mess up, you get yelled at. Possibly smacked. So don't mess up. And be patient with adults. They often think they know more than they really do. Most are not mathematically mature.

Let's imagine an equilateral triangle. "Equilateral" means it has three sides the same length. And you know what a triangle is. So start imagining one.

An Equilateral Triangle

Surprise! Quiz-Time:

15. Can you write down Euclid's Unbreakable Law and Euclid's Three Rules?

16. Infinity is a) an aardvark b) an additional annoyance c) an adverb.

17. And if you were German, how would you say "wow"? This is not something they say.

I was just making sure you were paying attention.

An equilateral triangle, which you were trying to imagine before I distracted you, has three equal sides. So imagine how one must look. It **must** look a certain way because its sides are equal. In our notation it is an eqSΔ and the capital S is for "sides." Don't make me explain the "eq" part. Or the "Δ".

18. Can you use your ruler to draw the eqSΔ you imagined? Break the law and measure the sides.

We can use Euclid to check both your imagination and your drawing. Here are his instructions.

19. Can you do this:

    ∀AB (a line, right?)
    ⊙A,AB × ⊙B,BA @ C,D

Join AC,BC
$\Delta CAB \equiv eqS\Delta$

The " ≡ " means "is" or "is equivalent to." It means you made precisely an eqSΔ because the two circles have same radius AB = BA and AC,BC are more radii (radius is Latin, plural is **radii**). So AB=AC=BC and these are the sides of ΔABC.

20. What do you get if, instead of joining AC,BC, you join AD,BD?

Now that you have an eqSΔ, is it as dreamy as you imagined it to be? More importantly, did your imagined triangle look like the real thing? What about the one you cheated on by measuring the sides? Which one turned out best?

A triangle has three sides, joined at their endpoints. When two lines meet, they make one or more **angles**.

21. How many angles does a triangle have? A square? A figure with six sides? With n sides? The "n" means "any counting, or whole, number" or in notation "$\forall n \in \mathbf{N}$" where "$\in$" means "in" and "**N**" means "all the counting numbers: (1, 2, 3, ...)."

22. Can you write a definition for your best idea of an angle? Use plain English. Don't leave anything out.

Now you should all read your definitions out-loud and vote on which one is the weirdest. "Angle" is like "straight line." Of course you know what it is. But you can't explain it without sounding stupid.

23. What do we call these absolutely clear ideas with stupid-sounding explanations?

We can do lots of things with an eqSΔ. But we need some more ideas first. Let's start with ...

Too Many Angles

The most important thing in pure geometry is your imagination. If you can't imagine it, you can't do it. Remember to begin always by imagining.

But many times you still may find something impossible to imagine. Sometimes this means you aren't imagining properly, that you are limiting yourself by false ideas. But sometimes it means that the thing you are unable to imagine is impossible to actually do. Learn to tell the difference. Ditch the false ideas.

So from this point, when I say "Can you draw ...." I am really saying "Can you imagine and then draw ...." But I am being lazy about it. Lazy can be good, as we learn in notation. But don't be too lazy to imagine.

24. Can you draw one angle with two lines?

Of course, you can. But don't be too lazy to use a ruler. All lines are straight. The lines here are called the **legs** of the angle.

Now look at your one angle, think about Euclid's Rule No. 2, and imagine turning your one angle into two angles.

25. Can you draw two angles with two lines?

26. Can you draw a right angle?

Of course, you can. A right angle is another ideal. Everyone knows it's the corner of a perfect square. An **acute** angle is less than a right angle (legs closer together) and an **obtuse** angle is more than a right angle (legs farther apart).

27. Can you make an acute angle?

28. What happens to your angle if you get too acute? In other words, what do the legs of the angle turn into?

We call this an angle of zero measure.

29. Can you make an obtuse angle?

30. What happens to your angle if you get too obtuse?

Now look back at the two angles you made with two lines. Write "acute" inside the acute one and "obtuse" inside the obtuse one.

31. If you add those two angles together, how many right angles do you have?

Some people call a straight line a "straight angle" in geometry. If you put two lines with a dot between them in a straight line, you do have an angle of 180° or 2∟. We are not going to call those two lines in one straight line an angle in this book.

But if we think of it that way, you can see that any two angles like yours add up to two right angles. These angles are next to each other so we call them

adjacent angles.

When two angles add up to two right angles, they are called **supplementary** angles. When two angles add up to one right angle, they are called **complementary** angles.

32. Can you make an angle and then make its supplement? What are their measure in degrees on a protractor?

Did you remember to simply extend one leg backwards in order to make the supplement?

33. What would you add to an angle to make a complement?

34. Can you make an angle and then make its complement? What are their measure in degrees?

35. Can you make a 30° angle and its complement and supplement using geometry and check them with a protractor? Hint: the angles of an eqSΔ equal 60°.

Just so you know, if you have two complementary angles, ∠A and ∠B, ∠A is the complement of ∠B and ∠B is the complement of ∠A. It works the same way for supplementary angles.

Quiz-Time:

36. Can you make two unequal adjacent angles with two lines? What kind of angles are they?

37. Can you make two equal adjacent angles with two lines? What kind of angles are they?

Well, that was easy. But we're supposed to be making too many angles.

38. Is it possible to make three angles with two lines?

If you used your pencil, you forgot to imagine first. Three angles with three lines is a Δ. Three angles with two lines is impossible.

Look at your two angles with two lines again and imagine first to answer this:

39. Can you make four angles with two lines?

40. Can you make more than four angles with two lines?

Did you imagine first on that last one? Let's talk about those four angles with two lines. The "⊥" means "perpendicular" or "at right angles."

41. Can you do this: AB⊥CD × CD @ E? Don't forget to label everything.

So we have: "AB, perpendicular to CD, intersects CD at E." This gives us four equal angles: ∠AEC ∠CEB ∠BED and ∠DEA each equal to a ∟ or:

∠AEC,BEC,BED,AED = ∟

We usually label angles as close to alphabetically as we can and use the same name each time. ∠CEB is ∠BEC but there's no reason to switch names all the time and confuse ourselves. So, two lines here make four angles. Each of them a right angle. Their sum is 4∟, right?

Imagine that you can move C closer to A, like a see-saw on its hinge, and try the next four questions without drawing anything:

42. Are any of the angles still equal? Which ones?

43. What do these four new angles add up to?

44. Which ones are acute? Which obtuse?

45. Which ones add up to two right angles?

These things are always true of any two lines creating four angles.

But wait! Is 4∟ an angle? It is, actually. And it looks just like 0∟. An angle of 4∟ takes you all the way around a circle. So you can save time by staying right there on the starting line and waiting for everyone else to catch up.

Let's do more angles with more lines.

46. Can you make three angles with three lines? What do you call what you just drew? If it's not a triangle, you used your imagination.

47. Can you make two angles out of three lines without turning two of them into one line? Put your pencil down.

Think triangle. Think about Rule No. 2.

48. With three lines can you make 4 angles?

49. With three lines can you make 5 angles?

50. With three lines can you make 6 angles in two

different ways?

This next one is a little harder but you have all the ideas you need for it. Imagine first, because some of these may be impossible.

51. With three lines, can you make 7, 8, 9, 10, 11, 12, 13 angles?

The next one is even harder but you have all the tools.

52. What is the greatest number of angles you can make with four lines?

Alright already! That's enough angles. Let's whack some lines up into tiny bits to release our frustration.

N-Secting a Line

Quiz-Time:

53. Can you make an eqSΔ?

54. Can you write the notation for what you just did?

Fine. Let's do this instead:

∀AB
⨀A,AB × ⨀B,BA @ C,D
CD × AB @ E

According to Euclid, E is now the "midpoint of AB" or:

E ×/2 AB

When we cut a line in two, we bisect it or "cut it into two equal parts" or "2-sect" it. Our notation for "bisect" is "×/2", combining the intersect sign with ½. Cutting into three equal parts is "trisect" and four is "quadrisect." But let's be lazy.

55. Can you write "trisect" and "quadrisect" in our notation?

Excellent laziness.

56. Can you do this: ∀AB, C ×/2 AB? In other words, use your tools to find point C that bisects AB. Make AB at least four inches long so we can reuse this diagram.

We don't have to label any points we don't need labelled. So in #56, we just use the intersections of the circles to make a line that cuts AB at C. You can be lazier than this: just draw the four little arcs of the circles that you need for making that line. We don't have to do any more work than we **need** to.

But we have to do **everything** which is geometrically necessary. Laziness has its limit.

57. Using your existing diagram of C ×/2 AB, can you ×/4 AB with points D and E?

If you used three vertical lines for C, D, and E, you are not being lazy enough. We use our imagination to save ourselves work.

58. If you have C and D on AB in your diagram, what is the laziest way to get E?

59. Using your new lazy method, here's a problem:

Given: ∀ AF

Required: AB = BC = CD = DE

Two things to learn here: First, read the whole problem before you draw anything. AF has to be long enough to cut out four pieces with a compass. Second, no one said how big AB had to be. So just mark a B so that it's easy to put your compass spike on B, pencil-end on A, and spin a compass for C. Then repeat for D and E.

Diagrams should always be big enough to work with and neat enough that you can read your own labels clearly. Immanuel Kant, who might have been even smarter than Wittgenstein, said:

Mathematics is the science of diagrams.

And he was right. Try doing all this in your head if you don't believe me.

We know now that we can divide a line into halves and halves of halves and halves of halves of halves and...(popsicle). But dividing them into odd numbered equal parts is another matter. And this includes trying to make 6 equal parts because 6 = 2 × 3 and 3 is odd. Before we can ×/3 AB, we have learn how to ×/2 ∀∠. But first:

60. Can you do: ∀AB, ×/4 AB @ C,D,E as above, and figure out how to make a square on the first quarter, AD, using only your compass and ruler? No fair using the corner of your ruler.

Whacking Up Angles

How many of you figured out how to really make a square in #60? How many of you almost did? How many of you ended up banging your head against your desk, yelling, "I hate geometry! I hate geometry!"?

Real mathematicians do all those things. Even the last one. Then they settle down, wander off, finally come back and solve it later.

With problems in a book, sometimes there are solutions to study. Study all the solutions you can -- but only after you try hard to solve things yourself. This book has no solutions because I know that you can solve everything here yourself, no matter who you are. Have faith in yourself, no matter what anyone else thinks about you. You **can** solve everything in this book.

So, let's whack an angle in half. First, we need ∀∠. A good ∠ for an ∀∠ is a little less than ½∟. Here are two ways to say "any angle with vertex A and legs AB and AC":

∀AB,AC ∀∠BAC

61. Can you do this? It's Euclid's way of bisecting an angle.

∀∠BAC

∀D,E∈AB,AC: AD=AE

(Line two means: "Any D on AB, any E on AC, such that AD equals AE." The ":" is read "such that." Use the lazy way for AD=AE.)

Join DE

(Look -- ΔADE is an **isosceles** triangle or "isosΔ" because it has two equal sides. But forget that for now.)

⊙D,DE × ⊙E,ED @ F,G

(G is on the bottom.)

Join AG

AG ×/2∠BAC

(Or "AG is the bisector of angle BAC")

62. If FG × ED @ H in that last diagram, what is H for ED?

63. How many things does AG cut in half?

Now think about what you used the compass for in that last diagram and imagine doing this:

64. Can you bisect an angle without using circles -- if you can use a ruler to measure things? Do it, if you can.

If we are going to run around whacking up angles, we should at least know what we are whacking.

Imagine that you, like Euclid, cannot measure an angle with numbers. There are no protractors in the universe.

65. Can you imagine a way to measure angles by comparing them with some other angle or angles?

Euclid's way to measure angles uses right angles. He, like ourselves, knew what a right angle was. And if he could compare any angle to a right angle in a meaningful way, he could measure it.

One of his tools for this was Book I, Proposition 32. Part of this says:

The three angles in any triangle add up to two right angles.

There is an easy way to prove this with parallel lines. I'll show it to you when we get there. But we are talking measuring angles here.

We have two ways to measure angles that both use the fact that four right angles take you around a circle. The first method uses **degrees** and there are **360°** in a circle. The "°" means "degrees." The second method uses **radians** and there are **2*π*** (two pi) radians in a circle. If you think about it, you can answer this next question without knowing what π is:

66. Can you say how many radians are in a right angle?

67. Can you say how many degrees are in a right angle?

Remembering that the angles of any Δ add up to 2∟:

68. Can you say how many right angles are in each angle of an eqSΔ? How many radians? How many degrees?

Do not make a big deal about π. If you have a circle of radius 1 inch, the circumference is π inches. How big is π? That's a good question. Calculating π is like drawing an infinite line.

Do not let this freak you out like it did the ancient Greeks. Whenever they found a number like this they totally freaked out and had to count backwards from five, like in Peg+Cat -- which I watch. I am not speaking down to you.

Calculating π is an adverb, a popsicle trap. But we can approximate π closely using 22/7 and even more closely with 355/113 and even more closely if we have to. But I would prefer not to.

Our 22/7 is 3.142857142857... and 355/113 is 3.141592920354.... Using 3.14159 is almost always good enough for π in this imperfect world because it is within one one-hundred-thousandths of our unit length (feet, inches, inch-worms, your choice).

69. How long is the standard international inch-worm?

Trick question. Back to angles. In Euclid, we can only use the angles we can make. Our eqSΔ gives us three angles of 2/3∟. The only way to get more angles here is to two-sect them.

70. If you ×/2 2/3∟ how many right angles, radians, and degrees are in the resulting half-angle?

71. If you bisect that last result, how many right angles, radians, and degrees do you have?

I can see you get the idea. We can construct only certain angles, then we can bisect them. If we add some of those results from two different angles together we can get more angles. But in Euclid, this is the only way we can get a measured angle. So if we want to use a right angle and its bisections, we have to be able to build a real right angle. Let's warm up first:

72. Can you make a 45° angle with a protractor and double it with a ruler and compass to a 90° angle? Think about how bisecting an angle works.

73. Can you make a 60° angle with a protractor, cut it in half and then add one of those halves to the original for a 90° angle?

74. If you think about all the tools you have, can you imagine a way to create a real right angle from scratch?

Really Right Angles

Stupid Quiz-Time:

75. Can you construct a 30° angle? How many right angles, degrees, and radians are in it? (Don't get the degrees wrong!) If you ×/4 it, what angle do you have in right angles, degrees, and radians?

76. Explain as clearly as you can why we can't build an angle of 2/9∟ (20°).

Here are Euclid's instructions for building a line "at right-angles to" or "a **perpendicular** on" line AB.

77. How many right angles will we have on either side of the perpendicular we build?

Work these instructions out in your head, without drawing a diagram, using your imagination. Concentrate:

∀AB ∀C∈AB
∀D∈AC
CE=DC
eqSΔFDE
Join CF
∴CF⊥AB
("Therefore, CF is perpendicular to AB.")

78. Can you draw the diagram for those last instructions? Did your imagination get it right?

If any of you think that I am repeatedly using "imagination" as some sort of bogus teaching tool, you are incorrect. David Hilbert, a profound mathematician, wrote a book called *Geometry and the Imagination*. And it is way over most people's heads. The imagination is a very powerful tool and you will never outgrow the need for it (unless you become really boring). Use it early and often. Pump it up.

79. What should you use in #78 to make CE = CD in a lazy fashion and why does this work?

This idea of using the equality of all the radii of a circle to find equal things is very useful. In the next question, the notation "E·|·(AB,CD)" means "E is between AB and CD" and "∉" means "not on" and "eqD" means "equidistant" or "equally distant from" or "same as much far-away-ness" if your English ain't so good.

80. Can you solve this problem:

Given: ∀AB,CD ∀E∉AB,CD: E·|·(AB,CD)

Required: F,G,H,J∈AB or CD: E eqD F,G,H,J

81. Does the solution to the last problem work if the lines are parallel? (You know what parallel is. Use the sides of your ruler.)

82. What if AB and CD are further apart than your compass can reach? Could you actually make a substitute compass? How?

83. What if AB⊥CD × CD @ K and E is inside ∠AKC? Does the solution still work?

84. If AB × CD @ K at any angle, what could happen so that you could get F,G but not H,J?

Okay, think about all these last few things we just did because it's Quiz-Time:

85. Can you solve this:

Given: ∀AB ∀C∉AB

Required: CD⊥AB

86. Can you write down the notation for what you just did to get your solution?

87. If you choose a point on a line, can you draw a perpendicular from the line at that point?

88. Can you draw a line AB with endpoints A,B and raise a perpendicular at B? Do not forget Euclid's rules. You can always use the rules.

89. Can you ×/2 ∟? What is the measure of the result in right angles, degrees, and radians?

90. Can you ×/4 ∟? What is its measure in right angles, degrees, and radians?

Now get your imagination ready. In Euclid, we can construct things, which means we can take any piece that obeys the rules and build something with it.

91. Can you imagine a way to 3-sect a ∟? Can you write the notation for your method?

Enough with the quiz.

92. Do this right quick:

∀AB CD⊥AB ×/2 AB @ D

∀E∈CD Join E[A,B]

Now, Euclid wouldn't prove things this way. But if you put the point of your compass on E and the pencil on A, the circle you make will fall on B if your work isn't too sloppy.

93. What does this tell you about EA,EB? What about DA,DB?

You remember I told you, back there somewhere, that that an isosceles triangle (isosΔ) has two equal sides.

94. How many equal bases does it have? Why do I mess with you like this?

95. Can you write the notation for creating an isosΔABC where A is the apex and BC is the base?

96. Is an eqSΔ an isosΔ?

97. Given an eqSΔABC, how many sides can be the base of it to make it an isosΔ?

Here is something we can use both bisected angles and perpendiculars on. Let's think about this next problem.

Can you find the center of an eqS∆ and put a circle in the triangle touching the sides and a circle around it touching the points A,B,C?

Let's start by imagining the circle around it. This is called a "circle described on ∆ABC."

98. So where does the center of the circle have to be? If the center is P, what can you say about what PA,PB,PC have to be?

This result is true for all triangles with a circle on them. All the radii of a ⊙ have to be the same length, which means that points A, B,C are on the circle (A,B,C∈⊙) as the endpoints of radii. We say the "center is equidistant from the vertices of the triangle." Now think about all your recent tools:

99. Given eqS∆ABC, how do we get all the points that are equidistant from A and B?

If you think about #92, ∀E∈CD is eqD A,B, right?

100. Can you give the instructions for finding the point in the middle of any ∆ABC that is eqD A,B,C? Draw a diagram if you need to. (You will need to.)

101. Let's say we find a point P that is eqD(A,B) and eqD(B,C). Is it also eqD(A,C)?

When something is like this we call it **transitive**. In plain arithmetic, if A=B and B=C then A=C. It is the **equality** of distances that makes this transitive quality work for finding our center. We can use it to help our laziness. You only have to prove two sets of points are equidistant from P to get all three sets. Watch for this in geometry.

102. Can you draw an eqSΔABC and put a ⊙ around it? Use all the things you just learned.

The easy notation for the circle described on a triangle is "en⊙" because it "encircles" the triangle. The circle in the triangle is called the "inscribed circle" or "in⊙." Not many people use this notation. But it works fine and you can spread the gospel of laziness by using them.

103. Using your imagination and a pencil to think with, what does the center of the in⊙ have to be eqD from?

104. Which of your tools will produce a line of points eqD from those?

105. Can you create an eqSΔ and its in⊙?

106. Why is the center of the en⊙ also the center of our in⊙ here?

I should point something out before we go on. The lines we have used to find the centers can be described in this way: "The perpendicular from the midpoint of AB is the **locus** of points which are equidistant from A and B. And the bisector of an angle is the locus of points equidistant from its legs.

These are the **loci** which concur at our centers."

I point this out because most books don't use "locus" or "loci" and when they do, they don't explain it. They just say: "Find the locus of ...", leaving you thinking: "Find the *what?* Will I need a magic wand for that?" I got all the way through a degree in mathematics at a major university without hearing the word "locus." But it's out there, lurking somewhere.

The reason I didn't put a number in front of "Can you find the center of an eqSΔ and put a circle in it and a circle around it?" was to show you that many problems need to be broken up into sub-problems. You break them up and think about one piece until you can solve it. Then you go to the next piece. Always start with the easiest piece and work up from there.

Final question. And do not totally freak out. You **can imagine** this -- so you **can do** this.

107. Can you solve this:

Given: ∀AB

Required: C,D ×/3 AB

We will talk more about trisection later. If no one solves this problem, no worries. I'm going to give you three ways to solve it by the time we're done.

Impossible Trisections

Here's the first way. You can build a trisected angle less than 2∟.

108. What are two angles that you can bisect so that three of the results added together are less than 2∟? What do three of each add up to?

In Euclid, you learn to move lines around and copy angles to any point. So you can stack three of your angles into a single trisected angle without breaking any rules. Some of you have already imagined this in the last chapter.

109. Can you say how you use this trisected angle to trisect your line exactly?

One way is to treat your angle like you were going to bisect it. You put a line across the angle exactly as in bisection of an angle.

110. Can you write the notation for the whole process of trisecting ∀AB?

What you end up with is a triangle with two lines in it which, when produced, trisect AB. Make sure you have such a diagram before we go on.

This is a good place to talk about impossible things.

It is impossible to trisect ∀∠. Looking at this trisected triangle you just made, it really looks like this method would trisect ∀∠. But you can only trisect angles that you can build. Here's why. We are going to talk about a little algebra. But you will understand the part that matters. We are just putting geometry in another language.

In Euclid, we can only make lines and circles or their arcs. You know that. In algebra, all lines can be put in this form:

$$ax + by + c = 0$$

The letters a, b, and c are numbers, x and y are variables. In algebra, all circles can take this form:

$$x^2 + y^2 = r^2$$

Here, x and y are still variables and r is the radius. Here is an important idea:

Any algebra problem solvable with line and circle forms can be solved by Euclidean geometry.

The pieces of the last two algebra equations are what we can do with a ruler and compass. But that's all we can do with a ruler and compass. If you want to trisect any angle, the algebra solution has a term with **x^3** in it. Its curve is **not an arc of a circle**. You can solve trisection with trigonometry, too. Again, the curve of solution is **not an arc of a circle**. So it is truly **impossible** to **directly construct a trisected angle with a ruler and compass** unless you can build it. But that is all the impossible it is.

111. Try this. Make ∀∠BAC. AD=AE like in 2-secting an ∠. But choose D so that E and D are maybe three inches apart. Join ED. Now **very carefully** 3-sect ED at F,G using the centimeter marks on your ruler.

What is interesting is that the Greeks, like Euclid, knew that this was impossible without measuring anything or using the not-yet-invented algebra or trigonometry. They invented the **cissoid curve** to solve trisections. How did they know this kind of triangle you have doesn't trisect its angle?

I will give you a hint. Put your compass point on the apex of the triangle and put a circular arc just inside the base of the triangle, all the way across. Join the points where the four angle lines cut the arc. Now you have three circular sectors with chords against their arcs.

112. Can you think of a lazy way to show that the three angles are unequal?

Now if your three chords are equal, you accidently drew a trisectable angle. But most of you found the center chord is bigger than the outside chords when you put the compass point on F, the pencil on G and swung it around to the outside. Euclid never says this directly, but angles are measured in Euclid by their arcs. His definition of angle is no help at all.

113. Can you draw a second big arc below the first one?

Here the angle is unchanged even though the circle is smaller.

114. Can you say why the same angle has the same measure if your two arcs are different sizes? This is the same thing as asking why a right angle on any circle center equals a right angle?

The answer is in the diagram you have not yet drawn: two circles, one center, one angle. Always -- a diagram.

I want to say one more thing about impossible things before we learn another way to 3-sect. It is impossible the **directly construct** the trisection of any angle with a ruler and compass. But our minds are **infinite**. The men and women of mankind have been thinking infinitely and they will never stop.

We came up with one real idea, then two, three .... How many real ideas has mankind had? We will keep thinking infinitely until we are all popsicles beneath a dead sun. Or maybe we'll get to a new sun before that and still be thinking infinitely.

And the unthinking people who have been pleased to tell us that something is impossible **have always been wrong so far.**

So someday, someone will think of an **indirect** way to trisect any angle with a ruler and compass. This solution will be very hard to think of. But one day it will occur to someone and it will be simple and elegant and many of us will wonder why we didn't think of it ourselves. This could happen tomorrow. Or after the sun burns out.

If you tell this to an adult and he says that it is wrong, you know what to do. Only this time, leave Wittgenstein out of it. Just say:

I don't think we should limit the possibilities of infinite thought.

Here is an easier way to ×/3∀AB:

115. Can you do this:

∀AB ∀AC⊥AB (make AC pretty short)
BD⊥AB: BD = 3AC and D opp. side AB from C
CD × AB @ E: AE = ⅓AB

116. Can you draw ∀AB and put E at 1/5?

117. Can you draw a line AB and put a C, D, and E at 1/2, 1/3, and 1/4? Can you be lazy about it? No measuring!

118. Can you draw a diagram that shows that 2 goes into 3 one and a half times?

119. Can you draw a dragon eating the people that tell us every good thing is impossible? Well, can you?

Some of you may be getting tired of triangles. I have bad news for you ...

Too Many Triangles

Quiz-Time:

120. Can you make an equilateral triangle? Does it make you want to scream? If you all do one short scream in unison, you can probably get away with it.

121. Can you make an isosceles triangle, measure its angles with a protractor and add them all together?

I should finally mention that when we label ∆ABC, the consistent way is to make A the apex, BC the base, putting B on the left. Not everyone does it this way. But consistency avoids confusion.

122. Can you write the notation for making ∀isos∆ABC? If you follow your own instructions, are the labels in the right place?

123. Can you make an isos∆ with a base of 1 (inches, miles, whatever) and the sum of sides 3 whatevers?

124. If a three-inch unit is a "thrinch," what is a two-inch unit?

Seriously, a base of 1 3-inch line and the sum of sides of 3 3-inch lines or 9 inches would be easy to work with in that last problem.

125. Why would using ⅓ inches as a base of 1 and a sum of 3 ⅓-inch lines work in that last problem? Would it be annoying to draw?

126. What do the angles in a triangle add up to, in right angles, degrees, and radians?

127. If you have the measure of an angle in right angles, how do you change it easily into radians?

Whew! That quiz was kind out of control. But it made a good review.

Besides eqSΔ and isosΔ, there are other common forms of Δ that we need to know. A **right triangle** or "⊿ABC" or "∟ΔABC" has a right angle. So our notation goes:

⊿ABC ∟A

128. Can a ⊿ can have an obtuse∠?

129. Can you make a right triangle using only geometry? Can you give the notation for your method? Can you measure its angles with a protractor and add them together? Do two of its angles equal the third? If so, why?

130. Do this to make a ⊿ABC from Book III in Euclid:

∀BC D ×/2 BC ⊙D,DB

∀A∈arcBC Join A[B,C]

∴ΔABC ≡ ⊿ABC ∟A

131. Can you use that last method to make a triangle with sides of 3, 4, and 5 inches? Imagine first! If you do one thing here, the method does the rest. What is that one thing?

In any ⊿ABC ∟C, if we label the sides opposite ∠A as a, opposite ∠B as b, and opposite ∠C as c, then if we measure the sides, they will satisfy this equation:

$$\mathbf{a^2 + b^2 = c^2}$$

If you haven't done algebra, a^2 is laziness for a×a. This doesn't seem like much laziness for a×a, but in algebra you can have a×a×a×a×a×a×a×a×a×a×a×a×a and writing a^{13} instead will keep you from losing your mind.

132. Can you make a right triangle with sides 6, 8, and 10 inches? Don't draw it. Think! How about 9, 12, 15?

133. A triangle with sides of 3, 4, and 5 is a right triangle. The next bigger one with integers for sides also has a side of 5. But its the smallest side. Can you come up with a strategy for finding the other two sides? If you can, does it work? What are they?

134. Can you make a right triangle with sides of 6, 8, and 9 inches? If not, why not? And if you start to use a ruler, spank your hand. Do the math.

It is **important** to realize when simply doing the math is easier than drawing a diagram.

Most triangles are not right triangles. An **acute triangle** has three acute angles or "all angles are less than ∟." "Acute triangle" is never a typo for "a cute triangle." No triangles are cute unless you put non-geometrical elements in them like anime eyes and Hello Kitty whiskers.

135. Can you draw a cute triangle? It's always good to take a break like this. Just don't be naughty.

136. Now can you make an acute triangle? Measure its angles with a protractor and add them together.

A triangle with an angle greater than a ∟ is an **obtuse triangle**.

137. Can any triangle have two obtuse angles? If not, why not?

138. Can you make an obtuse triangle? Measure its angles with a protractor and add them together.

Here's kind of a cool thing. Go back to your ⊿ in the ⊙ in #130.

139. If you put another ∆EBD in the bottom arcBC, what kind of ∆ must it be?

Now draw another circle and put BC about halfway up from where the last one was. Use the arcs, above and below, to create any two triangles.

140. What kind of triangle is the top one? And the one on the bottom?

Imagine starting with the first diagram, grabbing the diameter BC and slowly pushing it up to where it is in the second diagram.

141. What do you imagine is happening to the angles at the apexes of the triangles?

142. In the two diagrams of our two ∆s in those ⊙s, if we call the top angle ∠H and the bottom one ∠L, what does ∠H + ∠L equal in the first diagram? And in the second diagram? Use a protractor, if you need to.

In both our diagrams here, we have a figure of four sides. Euclid calls any such figure a **quadrilateral**. Figures with sides are called **polygons**. So I call a figure with four sides a 4-gon and save nine letters. Because the vertices -- those points where the sides of the 4-gon meet -- are on a circle, we call the 4-gon in the diagrams a **cyclic 4-gon**.

143. Can you draw a circle and draw ∀cyclic 4-gonABCD in it? When we label an n-gon (∀n∈**N** of sides) we start on the top or top left and go clockwise.

If you compare yours with other people's, you find some of them are very different. Let's see if that matters.

144. Join AC,BD which should give you two **diagonals**. If not, fix your labels. What do the angles on the opposite diagonals (∠A+∠C and ∠B+∠D) add up to?

145. So what do all the angles in your cyclic 4-gon add up to? Is this true of all your cyclic 4-gons?

Let's use our imagination to see if it is true of all 4-gons.

146. Can you imagine pulling one corner of the 4-gon out of the circle until the opposite angles do not add up to 2∟? Can you draw this 4-gon? What do you think all of its angles add up to?

The angles from the vertices of all 4-gons equal 4∟. But only in a cyclic 4-gon do the opposite angles sum to 2∟. In your diagrams, you can see why this is true.

Many things are true of all triangles, as well. So they are true of each and every triangle. Or ∀ΔABC. We call this an **any triangle** or **scalene** triangle. It **cannot** be an eqSΔ, an isosΔ, or a ⊿. If you make one of those, your diagram will mislead you in your work. Things will be true in your bogus ∀Δ that are not true in a real ∀Δ.

147. Can you make an ∀Δ, measure its angles with a protractor and add them together?

Some of you will want to do more geometry later. Some of you would rather volunteer to do dishes in the cafeteria and then mop the bathrooms. Anything but geometry. If you're going to do more geometry, here is a good ∀ΔABC to work with.

148. Can you do this:

AB: think of a clock face. A at 1. B at 6.
AC: hold AC⊥AB then bring it down a bit.
Join BC

If you now have an isosΔCAB, you brought AC down too far. For doing Euclid problems, you want the triangle three or four notebook lines tall.

Back to our triangles:

149. Can you make a triangle with sides of 2, 3, 5 inches? If not, why not? So what has to be true of any two sides of a triangle?

150. Can you make three lines, any two greater than the third, and then see how many triangles you can make with them?

151. If you have three lines, any two greater than the third, and make the longest line the base, how many different triangles can you make? (Do not draw them again.)

152. So how many actually different triangles can you make with three given sides?

Let's see if you are correct.

A lot of things in Euclid come from proving that two triangles are "equivalent" or " ≡ " which means "same in every way." For triangles, this means that the three sides are the same length in both triangles. Also, the angles **between the same sides** are the same.

There are three propositions in Euclid used to prove two triangles are equivalent. You remember that " ∀ " logically means "any" or "every" or "all." Logically, when you say two triangles are equivalent, you are saying that the two of them are one, and only one, triangle. Think about that until it seems okay.

Because it is not only okay, it's true. So the rules for showing two triangles are equivalent are the same rules for saying that if certain things are true, you have one and only one triangle possible for a triangle with those things.

The first rule says that if two triangles have a same (or equal) angle, and the legs of that angle are the same on both triangles (say, big one on top), then the triangles are equivalent: all angles and sides the same. And we can spin one around so it is oriented just like the other so they look the same. This is Euclid Book I, Proposition 4 or Euclid 1.4.

153. Why does 1.4 work? This is the same as asking: If I give you one angle with its two legs and you can't swap out the legs, why is only one triangle possible?

Euclid does not make it as plain as I have that he doesn't always allow us to flip triangles over. He will (logically) scootch one triangle around to put one on top of the other. But Euclid never flips one over.

Euclid 1.8 says that if two triangles have the same sides, they are equivalent.

154. If we talk about having the three sides making only one triangle possible, why does 1.8 work? Show your reasoning with a diagram. And no flipping.

Euclid I.26 says that if you have one equal side and any two equal angles, the triangles are equivalent. But this gives us two different cases. Either we have a side and an angle on each end or we have a side with an angle on one of its ends and another angle sitting out there all on its own.

155. Why does having one side with specific angles on each of its ends produce one and only one triangle? Can you draw a diagram for this?

156. And, finally, if we have a side with an angle on it, and another angle sitting out there by itself, humming a lonely little tune, why are we stuck with one and only one triangle? Do a diagram for this, too.

Between Parallel Lines

Euclid uses parallel lines to compare the area of figures. You have probably done problems of area before. But let's review:

157. Can you draw a rectangle that is two inches by three inches, making marks at the inches along the sides, and connect them so that you have a grid? How many squares are in the grid? What do we call each square?

158. A square has sides of six cm. Which rectangles can you make with integer sides that are the same size?

159. Can you make a rect∟, 9×4, and cut it into two parts that will make a square? This is actually cool and not too simple. Use some diagrams. I had to use two.

These are basic ideas:

1. We measure length using a unit measure.
2. We measure area using a unit square.

So if your lines are 2 inches and 3 inches, your rectangle is $2 \times 3 = 6$ inch^2 (square inches). Our unit measure is an inch or a centimeter or whatever we choose. The ancients Greeks had their unit measure.

But Euclid never defines a unit in geometry. For him, length and area can only be compared.

160. What does Euclid compare angles to?

There is nothing in length or area like that. So Euclid can only compare lengths and areas with each other.

161. Can you draw two rectangles next to each other and think of a way to compare their area without numbers?

162. Can you do the same thing with two triangles?

It is easy to compare things when they look different. But what if they look too much the same?

163. Can you take your ruler and, holding it carefully in the same place, use both edges to make lines across your paper?

These lines are parallel. But you knew that. And this is not the way Euclid makes parallel lines. We will do it Euclid's way in a little bit. First --

164. Can you draw two triangles, using the parallel (∥) lines you just made so that each triangle has the same size base (use your ruler) on the bottom line and its vertex on the top line? Make them as wildly different as you can.

I can tell you, from where I am sitting in the distant past, in a faraway land, writing this book, without looking at your paper, that your two triangles have equal area. So either *I am the eyes of Nostradamus and all your ways are known to me* or I am using Euclid's geometry. Which do you think it is?

Your two triangles are **equal** but not **equivalent**. When two things have the same area, they are equal. When they are the same in every way, they are equivalent. In notation:

$\Delta ABC = \Delta DEF$ means ΔABC is equal to ΔDEF

$\Delta ABC \equiv \Delta DEF$ means ΔABC is equivalent to ΔDEF

WARNING: some books use equal and equivalent the other way round in one way or another. So be alert and make sure you know which is about area and which is about everything in whatever book you are using.

We need to learn how Euclid made parallel lines because using both sides of the ruler is kind of cheating.

165. Can you do this:

$\forall BC \quad \forall A \notin BC \quad \forall D \in BC$ Join AD

Copy $\angle ADC$ to A for $\angle DAE$

(Copy the angle as best you can by sight.)

EA(pr) to F then $EF \parallel BC$

Now if EF does not look parallel to BC, it is because we are not using Euclid's exact method for copying angles. And if you use your imagination, you can see that your angles are off. Happily, now that we know Euclid's way to make $AB \parallel CD$, we can go back to using both sides of our rulers.

A parallelogram (‖gm) is a 4-gon with opposite parallel sides. Like this:

166. Draw ∀‖gmABCD: AB‖CD AD‖BC ∠A,B,C,D≠∟

167. Is a rectangle (rect∟) a ‖gm? Is a square? This means, are these things 4-gons with parallel opposite sides? The definition is **the definition**.

168. So which angles of a parallelogram have to be equal to each other?

169. Make two parallel lines. Make ∀∆ABC with A on the top line and BC on the bottom. Can you think of a way to build ‖gmABCD from ∆ABC?

This is a very useful thing in geometry, turning a ∆ into a ‖gm. Lots of things are true about a ‖gm that help you with the ∆. In notation, we could say, "‖gmize ∆ABC".

170. Join AC in the ‖gmABCD you just made. Can you say how much area ∆ABC is of all the area of ‖gmABCD?

Now really use your imagination for this next one. Draw some little helper diagrams if you need to.

171. Can you look at your ‖gmABCD and think of a way to turn it into a rect∟ of exactly the same size?

Let's see if your way is like my way.

172. Can you do this:

PQ∥RS ∀A∈PQ ∀B,C∈RS

Join AB,CA: ΔABC

CD∥AB × PQ @ D

CE⊥RS × PQ @ E BF⊥RS × PQ @ F

Now think about what we have. ∥gmABCD is 2×ΔABC.

173. Just by judging what looks equal in your ever-so-carefully drawn diagram of this, can you say whether Euclid 1.4, 1.8, or 1.26 proves ΔABC ≡ ΔCDA?

Don't worry if you got that one wrong. You don't have all the tools. Here's the answer: AC = AC on both triangles. Opposite angles of a ∥gm are equal or ∠ADC = ∠CBA. And alternate angles of a line between ∥ lines are equal or ∠DAC = ∠ACB. One side and two angles are good for 1.26. How close was your answer to this?

What we also have now are two other triangles: a solid ΔCED and an empty space of ΔBFA.

174. If ΔCED ≡ ΔBFA and we move the solid one into the empty space, what kind of 4-gon do we have?

175. Which of 1.4, 1.8, and 1.26 can tell us that ΔCED ≡ ΔBFA? You know that ∠F = ∠E = ∟ and that BF=CE between ∥lines. For 1.4, you need FA=ED. For 1.8, you need FA=ED and BA=CD. For 1.26, you need ∠B = ∠C or ∠A = ∠D. What do your think? And why?

In mathematics, it is good to think hard about something (for a little while) even if you don't quite have the tools. That way, your mind is ready for the solution when you do have the tools. So I won't give you the answer on that last problem.

If you will think back to the eyes of Nostradamus (that's from a great song by Al Stewart), you will remember I told you that your two very different triangles on equal bases between parallel lines were equal.

176. In that last diagram you made, no matter what ∆ABC looked like, everyone ends up with the same rect∟FBCE if what is true about their triangle?

The answer is actually in the labels ABC and FBCE. With equal bases, between parallel lines, all triangles are equal in **area**. Do not get confused and think they are magically somehow equal in sides or what we call **perimeter**, the sum of the sides. If you move the apex of one triangle down the parallel line towards infinity, that perimeter grows toward infinity. It gets very large. But the triangle's area does not change.

In the science of diagrams, trust in what you know and let that govern what you see. What you know is even more true than what you see unless your diagram is very carefully made.

A couple of last bits here.

177. In rect∟FBCE, if FB is 4 inches long and BC is 9 inches long, what is the area of FBCE? And if FB is height and BC is base, what is the formula for the area of a rect∟?

178. Does rect∟FBCE = ‖gmABCD back in that diagram (or have you been asleep for two days)? So what is the formula for the area of a ‖gm?

179. If you ‖gmize a Δ, how many of those Δ are in the ‖gm? So, what is the formula for the area of a triangle? Don't mess up here.

Let's talk about one more area before we go: the area of a circle.

180. Can you draw a circle, inscribe a square in it, and describe a square on it, making the diagram kind of an x-ray of a baloney sandwich on white bread with ketchup (also catsup or catchup from Cantonese kechiap meaning "brine of fish")? You learned something here today, didn't you?

Let's say the radius = 1 in this circle.

181. Can you say what the area of the outside square is? And the area of the inside square? How did you get your answers?

If we take the average of these two areas we get (2+4)/2 = 3. A circle of radius 1 something has an area of π something^2. So 3 something^2 is close to π

something² but not close enough for any test you'll ever take.

182. Can you imagine a way to get our two figures closer to the circle from each side?

183. Can you imagine a way to double the sides of the inside squareABCD?

Here's one way:

D ×/2 AB DH⊥AB × ⊙ @ H Join H[A,B]
Sym. rem. sides

That last bit is more laziness: "Symmetrically remaining sides." Or "do the same thing on the other sides of the square." When you finish you have an 8-gon inside the circle.

184. Can you imagine a way to double the sides of the outside square?

I'll describe a way. The circle touches AB of the outside squareABCD at its midpoint E. Same for midpoint BC at F and so on. So if you bisect arcEF and create a tangent line at that point, it will intersect AB and BC so that, doing this on all corners, you have a regular 8-gon outside the circle.

185. Can you make a sketch of how this works? Remember, a tangent line touches something at only one point.

Now you have 8-gons inside and outside the circle. The angled points of the inside one touch the circle and the sides of the outside one touch the circle just like the squares did, as tangents.

The areas of 8-gons are not hard to determine. But if I just give you the formula, you won't learn anything. So we'll stick to our imaginations.

186. Are the 8-gons closer to the circle than the squares? If they are, will their average area be closer to π?

Here's something you can do at home if you are **really** bored. Make the 8-gons into 16-gons, double again and again until you have 2408-gons. At that point, the average area is almost exactly 3.14159 and that is a good estimate of π.

187. Can you see that there is no end to doubling sides?

Calculating π is a popsicle trap. You can double sides until the sun goes out and you still have infinitely many more doublings to go. Do not calculate π. Other crazy people have already done it for you. Some, being over-trustful of computers, have probably taken it too far.

Oops, I almost forgot to show you how to use parallel lines to prove that the angles in a triangle add up to two right angles. I think I'll make you do the work.

Do this: PQ∥RS ∀A∈PQ ∀B∈RS Join AB

In Book I, we learn that ∠PAB = ∠SBA ∠QAB = ∠RBA because PQ∥RS.

Now pick ∀C∈RS, join AC and you have ∀ΔABC.

Using the equal-angle bit above and your other tools, why is ∠A + ∠B + ∠C = 2∟?

Mathematicians are just like you. When they get too many ideas in their head without really using them, they don't know which way is up. And then their head explodes. So, in the next part of the book, we will play with a bunch of the ideas that are bouncing around in your head and nail them down before someone gets hurt. But first --

Big Quiz-Time:

188. If you have a triangle with sides of 5, 12, and 13 and build squares on the sides, how many square inches are in the squares? Do the squares on the sides together equal the square on the hypotenuse? What kind of triangle is this?

189. Given a line and a point not on it, can you draw a line on the point parallel to the line using Euclid's method?

190. Can you show that two triangles, on equal bases BC,EF, between two parallel lines are equal? Do both diagram and notation.

191. Can you show that two triangles, on the same base BC, between two parallel lines are equal? Do a nice clear diagram and say where the notation would be different from #190.

192. Can you make two eqS∆ on the same base? One on top of the other is still only one in Euclid. Can you still somehow make two?

Playing Around

Think of this chapter as a very long and very enjoyable Quiz-Time. Or figure out a way to travel back in time to this distant land and stop me from writing it. Your choice.

193. Can you divide an eqS∆ into four equal triangles?

"That is absolutely not the only way to do that," said Nostradamus.

194. Can you think of another way to divide eqS∆ into four equal triangles?

195. Can you divide any triangle into four equal triangles?

Do not get lazy on this next one. The double-sized triangle has to be an eqS∆, as well. This keeps the problem from being trivial. It makes it a bit hard.

196. Given an eqS∆, can you make one twice the size?

You need to use your tools to think a little more abstractly about size. The size of any triangle of a given height (like between parallels) is half the size of a ∥gm that height which is equal to a rect∟ of that same height on the same base.

197. So if you take any Δ and turn it into a rect∟, how do you double the rect∟? And then how do you get the doubled triangle?

198. Remember when you put two eqSΔ on the same base? The 4-gon this makes is a **rhombus** and this rhombus is twice the eqSΔ. Is a rhombus a ∥gm? Can you use it to double the eqSΔ into another eqSΔ? If not, why not? And is that last question a red herring?

199. If you double the base of an eqSΔ, do you double the height? Can you show your answer in a fairly accurate diagram?

Tired of all these eqSΔ? Fine. Let's do some geometrical arithmetic.

200. Can you make a triangle that is equal to the sum of two other triangles between ∥ lines?

201. What if the two triangles are different heights? Can you still make the triangle of their sum? If not, what prevents you? Watch out for false ideas.

Let's say you have two rectangles with sides $a \times b > c \times d$. You need to add the little $c \times d$ to the big $a \times b$. Let's say $a < b$ and $c < d$. If we make length $r = a/c$ then $a = cr$. (Are you drawing a diagram of all this as I go along?) Then if we divide d by r, we have $(cr)(d/r) = c \times d$ and the rect∟ with side cr and width d/r can be added to $a \times b$ because cr equals a.

I say all this because there are tools in pure geometry that can take two lines, x and y, and create lines xy and x/y.

Assume you have these tools and try that last problem again. Then go to the next two problems, which are a bit harder.

202. Can you make a triangle that is equal to the difference of two other triangles between ‖ lines?

203. How about if the two triangles are of different heights? Can you still do it?

By the way, another way to say "height" in geometry is "altitude" and they mean exactly the same thing. Try your imagination, aided by little sketch diagrams, to answer this next question.

204. If you divide an eqSΔ with a line through its center ‖ to its base, how much of the triangle have you cut off on top?

Now think about all you have learned about figures between parallel lines for this next one:

205. If you divide ∀Δ with a line through its center ‖ to its base, how much of the triangle have you cut off? Can you ‖gmize it and show that you are correct using a diagram?

The next two problems are not trivial. See how far you can get on the first one. Then go look up the solution. It isn't hard to find but it will take some thought to understand it. You might not be ready to understand it. But that is okay. Try to get as much of a solution, to say what its pieces must be, how you might get them, and so on. Doing this is exactly what real mathematicians do. They **do not** always get a solution or even fully understand the problem -- until they do.

206. Can you divide an eqSΔ into two equal parts with a line parallel to the base?

If you understood the solution to the last problem, apply it to the next one. If you didn't understand it, don't just go through the motions to get some lame parallel line. The point is not the line. The point is your understanding.

Whenever you don't understand a solution, stick a bookmark in your mind, and come back later when you can understand it. There will always be problems that you are **not ready** to understand. So skip the next one if you're not ready.

207. Can you divide any triangle into two equal parts with a line parallel to the base?

When you do have that parallel line, you have a triangle above it and a **trapezoid** below it, which is a 4-gon with only two opposite parallel sides. Let's do some easier stuff.

208. If you know the base and height of a triangle, how many square inches is in it? Then what is the formula for the area of a triangle? Didn't I ask you this before? And you still forgot?

209. If there are 144 square inches in a square, what is the side?

$\forall \Delta ABC$ if the squares on the shorter sides add up to the square of the **hypotenuse**, or longest side, then the angle opposite the longest side is a right angle. You knew this. Let's notate this as

$$(a^2 + b^2 = c^2) \Rightarrow (\angle C = \llcorner)$$

210. If $a^2 + b^2 < c^2$, what kind of angle is $\angle C$?

211. And if $a^2 + b^2 > c^2$, what kind of angle is $\angle C$?

I want to show you that Euclid can be used to do lots of algebra. In algebra, you have things like $(a+b)^2$, $(a-b)^2$, and $(a+b)(a-b)$. In algebra we think of **a** and **b** as numbers. In Euclid, they are lines. Let's do letters, numbers and Euclid for $(a+b)^2$.

Letters: $(a+b)^2 = (a+b) \times (a+b)$

By the Distributive Law: $= a \times a + a \times b + b \times a + b \times b$

Simplifying: $= a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$

Numbers: $(4+3)^2 = (4+3) \times (4+3)$

$= 4 \times 4 + 4 \times 3 + 3 \times 4 + 3 \times 3$

$= 16 + 12 + 12 + 9 = 25 + 24 = 49$

Euclid: Do this with me.

212. $a = \forall AC$ $b = \forall CB$: $a > b$

$\therefore$ AC + CB = AB

Construct square AB^2 or squareABDE on AB

There you have $(a+b)^2$ in Euclid. So in this square we must have an $a^2 = AC^2$ which is a square on line AC , a b^2 or square on line CB, and two rectangles ab or with sides AC and CB, in notation AC•CB.

Let's continue with our diagram.

$AF \in AE$: AF = AC

FG∥AB × BD @ G

CH∥BD × FG,ED @ I,H

In this diagram, what are the vertices or corners of a^2, b^2 and each of the two ab?

Before I do more Euclid algebra, I want to show you two uses of $(a+b)^2$. The first one is how to calculate the square of a number in your head.

213. Let's say we have $(a+b)^2 = a^2 + 2ab + b^2$ and we want to square 12. Can you see a way to fit 12 into the algebra? Can you see the best way?

The best way is to make 12 into 10 + 2. Then we have $10^2 + 2\times(10\times2) + 2^2 = 12^2$. The easiest way to do the square in your head is to add $10^2 + 2^2$ and hang onto it. This is 100 + 4 or 104. If you had 7+5 = 12, you would have a=7, b=5 and would have 49 + 25 to add in your head. But 100 plus anything less than

100, like 4, is "100 and 4". So you hang onto 104 and do 2ab or $2 \times 10 \times 2$, however is easiest for you, and you get 40. 104 and 40 is $144 = 12^2$.

214. In your head, without using the pencil for anything but for writing down the answers, can you use this method to find 13^2, 16^2, and 17^2?

We can use this method backwards to find square roots of integers that are perfect squares, like 36 but not like 37. Let's say we have 121 and want the square root. We need an **a** and a **b** so that

$$(a+b)^2 = 121 = a^2 + 2ab + b^2$$

We want the biggest single digit which squared is less than the first digit of our number 121. Obviously, that is 1. So a=10.

215. Can you say why a = 10 and not a = 1? Think about how you just learned to square numbers.

This gives us $10^2 + 2 \times 10 \times b + b^2 = 121$ or $20b + b^2 = 21$. If you have enough algebra, you can solve the quadratic equation $b^2 + 20b - 21 = 0$ and use the positive root for b. But that seems a bit much for headwork.

There is an easier way. This is a shortened version of a method for computing the square roots of any number using your mind and pencil.

We start with 121 and the biggest square of a digit that subtracts from 1 is 1^2. So the first digit of our root is 1. We write down 1 (that 10 as **a**) and subtract $a^2 = 10^2 = 100$ from 121. This leaves 21.

Now we double the 1 for a tens digit of **2** in what follows. After the 1(a=10) of the root so far, we need a b×**2**b (b is a digit 1, 2, 3, ...) where b×**2**b ≤ 21 but (b+1)×**2**(b+1) > 21. Obviously 1×21 ≤ 21. This b=1 gives us the next digit in our root and the root is 11.

If any of this seems unclear. Go back over it again and use a pencil and paper to spell it out. When you do that, arranging it in a way that makes sense to **you**, you have an "algebraic" diagram.

Let's do $\sqrt{729}$. $2^2 = 4 < 7$. $3^2 = 9 > 7$. So first digit of root is 2. 2×10=20. $729 - 20^2 = 329$. Now we double 2 to 4 and need b×4b ≤ 329 where the b+1 thing still applies. 6 × 46 = 276. 7 × 47 = 329. So second digit is 7 and $\sqrt{729} = 27$.

216. Can you see that in these examples we are getting a (10, 20) and b (1, 7) for $(a+b)^2$? So can you see where the 2ab is in this method?

217. Is finding the square root of 121 ($\sqrt{121}$) the same thing as have 121 square inches in a square and finding the side?

On the next question, use our last method, combined with all the pencil or calculator help you need for the b×2ab search.

218. Using our $a^2 + 2ab + b^2$, if there are 169, 225, or 324 square inches in a square, what are the sides?

You might ask, "Why get a square root by hand when I can use a calculator?" Because scientists are already teaching ourang-outangs to use a calculator and those monkeys will work for food. I'm not joking.

You need to learn to use your mind so that you aren't competing with monkeys for a job. Calculators do **nothing** for your mind. Only thinking develops the only mind you have. And thinking makes everything easier.

219. If you square any number, what digits can the square end in? (There are five answers.)

220. If you square a two digit number, how many digits can you get? (There are two answers.)

221. What is the smallest two-digit number you can square and still get the smallest of the answers in #220?

Did you start with 30 in the last one and then work up? If not, you forgot to imagine. Imagining **is** thinking. Start with the a of a+b where a is the tens digit and you need the biggest one that doesn't bust 1000, right? 30 × 30 = 900. Squaring tens is easy and lazy, a mathematician's dream.

You have to imagine the workings of the problem and then the possible solutions. Imagination is like a whiteboard in the mind.

222. Using only your mind, with everything you have learned, what is the square root of 784? You can write down the intermediate values you get, if you need to.

223. Do some more thinking and explain how the idea of $(a+b)^2$, where you add two numbers and square them, fits into its algebra, its geometry, finding squares, and finding square roots? Make a table of the pieces at the top, those last four things in a left-hand column, and put how they fit together down the rows.

The four things that express the idea are some of the forms this idea takes.

224. What is a geometrical form of the expression:

$$a + b > c$$

In other words, what geometrical figure or figures express this idea? How does geometry express the idea?

These **forms** of $(a+b)^2$ -- and the next two ideas' forms as well -- will follow you everywhere in mathematics. When you can see how an idea expresses itself -- and that is what you have been doing -- you will recognize it in any form whenever it pops up. In order to recognize and understand any form, you must play with it like our playing with this idea of $(a+b)^2$.

You don't really understand an idea unless you can play with it in all of its forms which you can think of. You can see where imagination fits into this.

Let's do more geometric algebra.

225. Can you use the example of the algebra for $(a+b)^2$ to calculate the value of $(a-b)^2$ which is $(a-b)\times(a-b)$ in its algebra form?

The first time you do such a thing, you can just mimic the pieces like a monkey would.

226. Can you diagram this geometric version of the same thing like this?

a=AB b=CB ∴ a-b = AC

∀AB C∈AB: CB = b

AB^2 = squareABKF

CG∥AF × HE∥AB @ D: G,H∈FK,BK

EF^2 = 4-gonEFLI

Here's why we use notation. The next to last line is "CG, parallel to AF, intersects HE, parallel to AB at D, such that G is on FK and H is on BK." I like the short version.

227. Can you give the vertices for a^2, b^2, $(a-b)^2$, and the two rect∟ = 2ab in this diagram?

Our last geometric algebra is $(a+b)(a-b) = a^2 - b^2$. In geometry, a+b is the sum of two lines and a-b is the difference of two lines. So, in Euclid, this is the rectangle of the sum and difference of two lines.

228. Can you do the algebra for (a+b)(a-b)?

229. Can you make its diagram like this?

a=AB b=BC

∀AB C∈AB: CB = b (just like last time)

AB^2 = squareABKF

AB(pr) to I: BI=BC

AE∈AF = AC

CG∥BK × FK @ G

rect∟AILE: LE x CG,BK @ D,H

230. a^2-b^2 is a 6-gon with only ∟. Can you identify it by its vertices? So what are the vertices of a^2? And what are the vertices of the subtracted b^2?

This product of the sum and difference appears fairly often in Euclid problems. And it is often found in algebra. It is an important idea in either form. It is a part of several of the propositions of Book II.

Euclid goes further into algebra than many people think. Euclid 2.11 is the solution of

$$x^2 + ax - a^2 = 0$$

and geometrically solves both roots, if you extend the proposition slightly. But let's not go there. We've done enough algebra. Let's do something similar.

Similar to What?

Quiz-Time: Let's review the three rules for equivalent triangles:

231. What is the difference between equivalent and equal?

232. What has to be the same in Euclid 1.4 for the triangles to be the same?

233. What has to be the same in Euclid 1.8?

234. What are the two ways things can be the same in Euclid 1.26?

Some of you are wondering why triangles with three equal angles are not the same. Let's find out.

235. Quick diagram:

$\forall \Delta ABC \quad \forall DE \| BC \times AB,AC$ @ D,E

Now we have two triangles: ΔABC and ΔADE.

236. Are the angles in these two triangles the same or different?

Triangles which have the same angles are equiangular (eq∠). They are not necessarily equivalent. But they are **similar**. In triangles, if they are eq∠ they are similar. One notation for similar is this:

$$\Delta ABC \sim \Delta DEF$$

Some books use other symbols. Stay alert when dealing with strange geometry books.

When Euclid talks about equivalent and similar triangles or n-gons, he says it is okay if you move them around or rotate them to compare them. But you are not allowed to flip them over. If you folded a piece of paper in half and cut two triangles out of it at once, you would have two equivalent triangles. If you flip one over, it is no longer the same triangle in Euclid even though he agrees they are equivalent.

In this book, you can flip them over now all you want. If two triangles are the same or similar, even though one is flipped over, we will still call them same or similar as the case may be. Modern geometers flip stuff all the time.

237. Can you take a line and draw two equal triangles on it, oriented the same way? Can you make the triangles similar but not equal on another line?

Given the first line, you could have added a parallel line above it for equality. Remember your tools for creating equal triangles.

238. Can you divide ⊿ABC ∟A into two similar triangles? Will your method work for any triangle?

239. What kind of triangles do you have if you add AD⊥BC × BC @ D in our right triangle here?

240. Are all right triangles similar? This means equiangular, right?

Approach these next problems by using your tools in your imagination and then using a diagram to validate your approach.

241. Can you divide an eqSΔ into two equal parts? Into eight equal and similar parts?

242. Can you divide an isosΔ into two equal and similar Δs? Into two equal and not similar Δs?

Use your imagination to create a process for this next problem.

243. Can you make one triangle similar to another but twice its size? Before you diagram it, how many steps does it take? After you diagram it, was the process as you imagined it?

You might think that understanding a problem and then solving it is the whole process. But thinking about your solution and refining your solution, if possible, are more important. Otherwise, your hard work just slides out of your ear and onto the floor, where it lies forgotten. Let's use these same ideas with some 4-gons.

244. How many ways can you divide a square into two equal and similar parts?

245. Can you divide a square with one line into two equal but not similar parts? If not, why not? How can you divide a square into two equal but not similar parts?

246. How many ways can you divide a square into four equal parts and what kind of figures are the parts?

247. How many ways can you divide a rectangle into four equal parts and what are the parts?

248. How many ways can you divide a parallelogram into four equal parts and what are the parts?

249. How many ways can you divide a regular 6-gon into two equal and similar parts? How about for a 5-gon? Label the vertices to make it easy to write these answers.

This would be a good place to talk about dividing n-gons into triangles. Euclid proves that any two similar n-gons can be divided into equal and similar triangles.

250. What is the minimum number of triangles which make up a 4-gon? In which kinds of 4-gons are the triangles equal? Similar?

251. What is the minimum number of triangles which make up a 5-gon? A 6-gon?

252. So what is the minimum number of triangles which any n-gon? What are the vertices of these triangles that make up every n-gon in a similar way?

There's a lesson in that last problem. When working with n-gons, if your method doesn't work for all n, there is a good chance that you are wrong in your approach. Let's think about similarity a little more.

253. What would it mean for two rectangles to be similar?

Similarity, in Euclid, comes in Book VI. If you have two similar triangles, ΔABC ~ ΔDEF, then the sides on equal angles have this relation (remember that side **a** is opposite ∠A): a/b = d/e.

254. Which angles were those sides around?

This relation is true of the sides around any matching angle of similar triangles. In Euclid, **a** and **b** are lines. But this works with numbers, too. And it works with similar n-gons of any n. So if rect∟ABCD is similar to rect∟EFGH then AB/DA = EF/HE, these sides being on matching angles.

255. Can you take a rectangle and make a similar one?

256. Can you think of an easy way to make a similar rect∟ inside of a given rect∟?

257. Can you make a rect∟ equal and similar to that last result?

If that last one isn't the same as the given one, you messed up. If something is equal and similar, it is equivalent. Think about it.

258. If you have two similar rectangles, can you make a similar rectangle equal to their sum?

If I were a cruel man, we'd do the difference of two similar rectangles. It's tempting. But let's move on. I have a promise to keep.

Before I keep it, try combining imagination and tools to do the next two problems. Here is a possibly helpful idea: a square of sides 1 gives you a diagonal of $\sqrt{2}$; a rectangle of sides 3 and 4 gives you a diagonal of 5.

259. Can you construct a square with a diagonal of 3 and calculate the area? Use a calculator if you want to.

260. Can you calculate the area of an eqS∆ with a side of 1?

261. Do you wish you'd never heard of an eqS∆?

Another N-Section

Actually, this is the last of the three n-sections I promised you. But first,

Quick Quiz-Time:

262. What are the two ways to ×/n a line that you already know? Just describe them.

263. ∀AB, ×/3 AB @ C,D using an angle.

264. ∀AB, ×/5 AB @ C,D,E,F using a trapezoid and remember to do D, E, and F the easy way.

The third way to n-sect a line is by using parallels, which as we know from similar n-gons, divide things proportionally, which means "into equal fractions" of parts.

265. Can you do this ×/3 with ∥ lines:

∀AB ∀AC (make it long)
D∈AC: AD (< 1/3 AC. Just eyeball it.)
DE = AD EF = DE (using the compass here)
Join FB EG∥FB DH∥EG
∴ AB ×/3 @ G,H

You can see that you made line BF so that it was divided into three equal parts. By connecting FB, you could come back down and divide AB proportionally with those parallel lines.

266. Can you divide a line into 5 equal parts?

267. Can you divide a line into 3½ equal parts? Do this simply in two larger steps, if you can? What are the steps?

268. Can you divide a line just like another line is divided? This means to take some AB and divide it into a few random parts and then divide another line into the same proportional parts or "similar parts."

We can use these ideas for more things than n-secting a line. The next two problems can be done using these tools. But if you can imagine easier ways, go your own lazy way. You won't hurt my feelings. Just use Euclid's tools.

269. Can you show with a diagram that 1/2 + 1/3 + 1/6 = 1?

270. Can you draw a figure that shows that 3 goes into 5 one and two-thirds times? Does this mean that 5 is 5/3 of 3?

There's no reason to wear ourselves out on this third idea of n-secting lines. Just stick it in your toolbelt.

N-gons in a Circle

Let's warm up with Quiz-Time:

271. Can you make an isosceles triangle using a circle?

You remember that the easiest way to make two equal lines, if they are connected on one end, is to swipe them with a compass. Did you do that in the last problem by swiping the circle? If not, was your way lazier?

272. Can you make an isosceles triangle whose sides are twice the base? Do this the laziest way you can.

273. Can you reduce a 5-gon into a triangle and a rectangle that together equal the 5-gon? Do not make any extra work for yourself here.

Book III of Euclid is about circles. That's where the cyclic n-gons come from. Mostly, Euclid does triangles in circles. He loves triangles. But in Book IV, he puts every n-gon he can manage into a circle because n-gons are just glued-together triangles. Book IV is kind of boring. We will stick with what we know from Book I, put n-gons in circles, and try to skip the boring part.

274. Can you make an equilateral triangle using a protractor?

275. Can you imagine a way to make that equilateral triangle using a circle, as well?

Let's see if your way is my way. Mark a center on a blue line of your notebook paper, make a circle, 4 or 5 lines in diameter on that center, and add the diameter on that blue line, produced both ways a bit. Now put your protractor on the line, centered on the center of the circle and mark 90° on top. Flip the protractor over and mark 30° and 150°. Now connect the center of the circle to those three measured degree-marks. Those three lines will intersect the circle at the vertices of an eqSΔ. Connect the dots and done.

276. Can you explain where the numbers 90, 30, and 150 come from in those instructions?

277. Can you explain why we used the circle for the vertices instead of connecting the protractor marks?

When you do something several times and are paying attention to what you are doing, you come up with easier and easier methods. See if you can do that with the next five problems.

You want to find a method that produces a nice-looking regular n-gon. You have all the tools. Ready, set, ...

278. Can you make a regular 5-gon (pentagon) in a circle?

279. Can you make a regular 6-gon (hexagon) in a circle?

280. Can you make a regular 8-gon (octagon) in a circle?

281. Can you make a regular 10-gon (decagon) in a circle?

282. How about a regular 12-gon (dodecagon) in a circle?

Here's another way for some n-gons?

283. Can you make a regular 10-gon (decagon) from a regular 5-gon in a circle? What other n-gons, n < 20, can you make this way?

Some of you probably made these n-gons using a protractor and a circle. Some of you made them by making one n-gon in a circle and then doubling the sides by bisecting one and building as isosceles triangle on the side to touch the circle.

284. Which was faster? Which had a nicer-looking result? Or is it the person doing it who determines if something is faster or better done?

285. Can we make a regular n-gon in a circle of any n-sides or are we stuck with the angles we can build up through construction, like before, to determine the sides? No diagram needed.

286. Why can't we build a regular 9-gon in a circle?

287. Can we build a regular 15-gon (quindecagon) in a circle or is the angle a problem?

288. Can you do this:

Draw a nice big (don't get crazy) circle.
Put an eqSΔABC in it with vertex at 90°
Put a reg. 5-gonDEFGH in it, with D on A.

289. How many sides of a 15-gon between A and B?

290. How many sides of a 15-gon between D and E?

291. Then how many sides of a 15-gon are between B and E? Can you write the directions for creating a regular 15-gon in a circle now?

292. When it came to making the last 14 sides, did you make one, and then swing your compass round-and-round? If not, do you have a problem with laziness?

Let's learn something useful about angles in a circle:

293. Quick diagram:

SquareABCD: AC×BD @ O
en⊙O,OA
E ×/2 AB EF⊥AB × ⊙ @ F Join F[A,B]

294. What is the measure of ∠AOB in degrees, radians, and right angles?

295. Trick Question: What are those measures for ∠AFB? Why is this a trick question?

296. Add this: FE(pr) to G × ⊙ @ G Join G[A,B]
Now what is the sum of ∠AFB + ∠AGB?

The piece you are missing in all this is that angles on center (∠AOB) are twice as big as angles on circle (∠AGB). Both are on chord (line connecting points on circle) AB.

297. Now, what are the three measures for ∠AFB,AGB?

This shows you how the cyclic 4-gon fits into this picture.

298. One more piece: ∀H∈arcGC Join H[A,B]
What do you think the measures of ∠AHB are? You still have a cyclic 4-gon with the same diagonal AB.

299. So to finish up, take just the cyclic 4-gon part. Quick sketch of

⊙O, ∀chordAB, ∀C,D on arcsACB,ADB

What are the rules for the values of ∠ACB,ADB? And why can only one of them be the half of the angle on center?

Of course, if AB is the diameter, the angle on center is 2∟ and both angles on AB are 1∟. But that is the extreme case. Next idea:

300. Quick diagram: draw ⊙O; using the center and protractor, draw ∠AOB = 72°: AB ≡ chordAB, swing your compass do-si-do around the circle using AB to give us a regular 5-gon.

This 5-gon is called **convex** because all the angles stick out away from the center. Most n-gons we deal with, regular or not, are convex.

301. Flip one of your angles into the 5-gon, erasing the original, but (more or less) keeping the same measure of the angle. Is the 5-gon still regular?

The angle you created is called a **re-entrant** angle. Some things are true about all n-gons. Most seem to be true only of convex ones. Get ready to doodle.

302. What is the smallest n for an n-gon with one re-entrant angle? For two? For three?

You now know everything I have encountered about re-entrant angles. They don't show up much unless you go looking for them -- and I don't.

Draw a quick square and join the midpoints of the sides to give you four squares. Be lazy and quick.

303. How many squares can touch at a point?

When you join n-gons at a point and they cover the space around the point without leaving gaps or overlapping each other, they **tile**.

304. Do squares tile? OR Have you ever looked at the bathroom wall at school?

Let's play with these ideas just a little bit to see what is at stake behind these questions (not the bathroom one). It is easiest in a class room if everyone cuts an equal eqS∆, 6-gon, and 8-gon out of paper. Or do it any other harder way you can think of. You're doing all the work.

305. How many eqS∆ can touch at one point? Will they tile?

306. How many 6-gons can touch at one point? Will they tile?

307. How many 8-gons can touch at one point? Will they tile?

Here's one more idea from solid geometry that is useful and can be easily seen from the four tiled squares. Imaginations on, please.

308. Let's call the intersection, in the middle, point O. What happens to O, physically, in space, right there in front of you, if the angles on O are made any less than their current ∟ in those four squares?

Four right angles are 360°. Divide by three and we have 3 120° angles.

309. Draw an eqS∆ in a ⊙ and use it to make three lines on center 120° apart? Skip all the bits you don't need. What happens now to the center of the circle if we decrease the angles at all?

A **solid angle** is where **planes** (completely flat things that can only be stupidly defined) come together at a point.

310. So a plane solid angle is always less than what angle? Otherwise, how many planes have you got?

Okay, we're almost done. No more quick Quiz-Times. Let's do one more short chapter and get out of here.

Shaolin Geometry

These last problems are like the 36 Chambers of Shaolin, except we're 29 chambers short. But you've suffered enough, so we'll ignore the shortfall. You won't get any help from me because you don't need any. Do everything the easiest way that you can, without measuring anything with ruler or compass. Obey Euclid's rules. Use Euclid's tools. Just work your way through these problems, wearing your pajamas like a good Shaolin monk, and you're done.

I'll be waiting outside the temple, keeping the fire going, so you can burn those cool dragon and tiger tattoos onto your forearms as soon as you're done. You can get the optional ruler and compass tattoos but they are nowhere near as cool. We also have t-shirts and hoodies with Jackie Chan as Euclid.

With all that you have learned, you should survive any encounters you have with the real Euclid down the road. And don't think you haven't learned a great deal. You have not learned how to do proofs. But you have learned almost all the meaningful ideas from Euclid's Book I and then some. Good luck, Grasshopper.

311. Can you make a square that is the sum of two other squares?

312. Can you make a square that is the difference of two other squares?

313. Can you make a right triangle that is equal to a square?

314. Can you take a square and turn it into an equal triangle?

315. Can you make a rectangle that is equal to the sum of two other rectangles?

316. Can you reduce any 5-gon with a re-entrant angle to a triangle?

317. Can you make a square that is equal to three quarters of another square?

And that's all there is to pre-Euclid. You are fully equipped to handle the real Euclid now. Continue to think infinitely and you'll be fine.