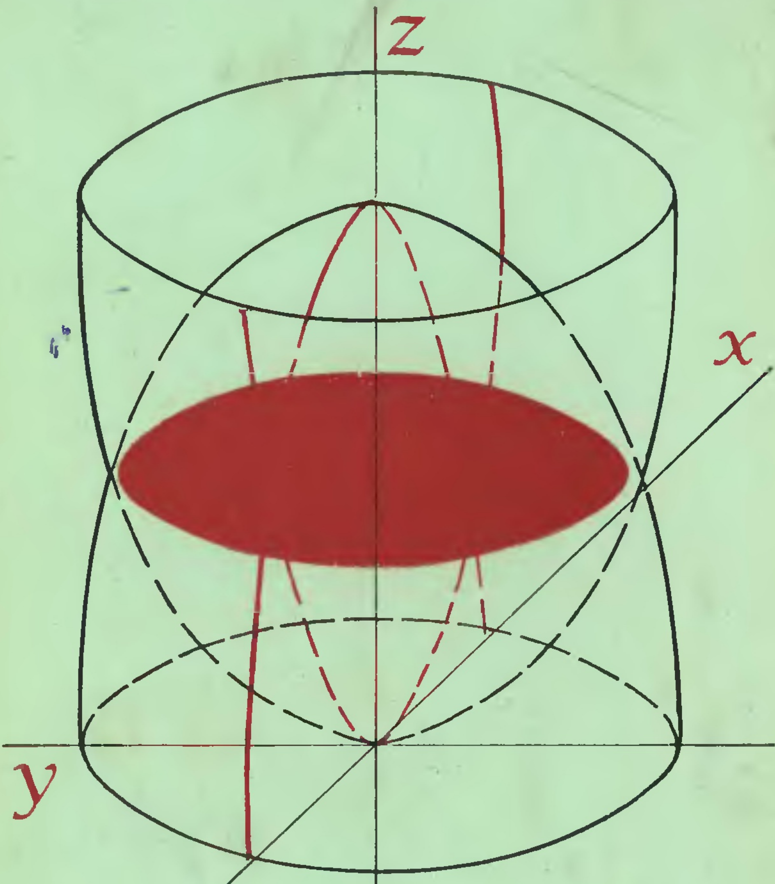
A. Pogorelov

GEOMETRY



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ABOUT THE BOOK

This is a manual for the students of universities and teachers' training colleges. Containing the compulsory course of geometry, its particular impact is on elementary topics. The book is, therefore, aimed at professional training of the school or university teacher-to-be. The first part, analytic geometry, is easy to assimilate, and actually reduced to acquiring skills in applying algebraic methods to elementary geometry.

The second part, differential geometry, contains the basics of the theory of curves and surfaces.

The third part, foundations of geometry, is original.

The fourth part is devoted to certain topics of elementary geometry.

The book as a whole must interest the reader in school or university teacher's profession.

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А. В. Погорелов

ГЕОМЕТРИЯ

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CONTENTS

PREFACE

Containing the compulsory course of geometry, this textbook is a manual for students of universities and teachers' training colleges. It is particular for the impact on elementary geometry and, therefore, highly professional training of the school or university teacher-to-be.

The geometry course is organically related to the school treatment of the subject. Starting with the study of coordinates and vectors, which is just a pleasant revision for the well-trained first-year student, the first part of the book, analytic geometry, is easy to assimilate, and actually reduced to acquiring skills in applying the coordinate and vector algebra methods to the solution of elementary geometric problems.

The second part, differential geometry, contains the basic facts from the theory of curves and surfaces.

The third part of the book, foundations of geometry, is original. In contrast to the traditional courses where the principal topics related to the axiomatic construction of geometry are solved on the basis of D. Hilbert's (or H. Weyl's) axiomatics, the treatment of the material is supported by the axiom system given at school. Thus, the problems of consistency, completeness and independence of axioms are solved with relation to the well-known axiomatics whose investigation is of undoubtedly professional interest. This part ends in the account of the basics of projective geometry in its analytic interpretation.

The fourth part of the book is devoted to certain topics of elementary geometry including those treated at school incompletely such as geometrical constructions, the measurement of lengths, areas or volumes.

Considering the book as a whole, it can be said to begin with the school treatment, and to return to it at a higher level, providing extensive and profound knowledge of the school subject. We believe that this will necessarily interest the reader in the school or university teacher's profession.

I take the chance to express profound gratitude to my co-workers Yu.A. Aminov, A.I. Medyanik, A.D. Milk and Yu.S. Slobodyan for useful criticisms and valuable assistance.

Part One

ANALYTIC GEOMETRY

Chapter I

RECTANGULAR CARTESIAN COORDINATES IN THE PLANE

1. Introducing Coordinates in the Plane

Let us draw in the plane two mutually perpendicular lines Ox and Oy termed the *coordinate axes* (Fig. 1) which intersect at point O called the *origin of coordinates* or simply the *origin*. The origin divides each of the axes into two semi-axes: a positive semi-axis shown by an arrow in the drawing, and a negative semi-axis.

Any point A in a plane is specified by an ordered pair of real numbers—called the *coordinates of the point*—the x-coordinate (abscissa) and y-coordinate (ordinate) according to the following rule.

Through the point A we draw a straight line parallel to the *axis of ordinates* (Oy) to intersect the *axis of abscissas* (Ox) at some point A_x

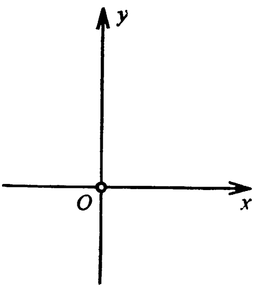

Fig. 1

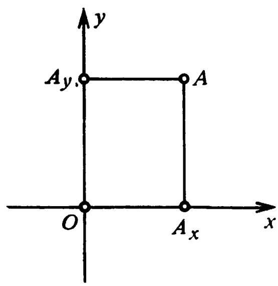

Fig. 2

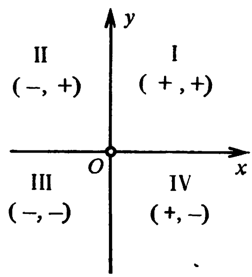

Fig. 3

(Fig. 2). By the abscissa of the point A we should understand a number x whose absolute value is equal to the distance from O to A_x being positive if A_x lies to the right of the origin and negative if A_x lies to the left of the origin. If the point A_x coincides with the origin, then we put the x-coordinate equal to zero.

The y-ordinate of the point A is determined in a similar way.

The coordinates of the point A are always enclosed in parentheses, $A\ (x,\ y)$.

The coordinate axes divide the plane into four right angles—*quadrants* I, II, III and IV (Fig. 3). Within one quadrant the signs of both coordinates are as shown in the figure.

Points lying on the x-axis (i.e. on the axis of abscissas) have y-coordinates equal to zero and points lying on the y-axis (i.e. on the axis of ordinates) have x-coordinates equal to zero. The origin of the x-axis and the y-axis has zero coordinates.

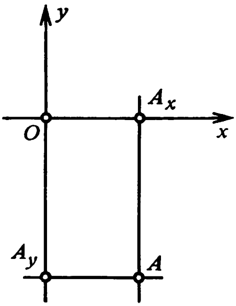

Fig. 4

The plane on which the x- and y-coordinates are introduced by the above method is called the xy-plane. An arbitrary point in this plane with the coordinates x and y will sometimes be denoted simply as (x, y).

For an arbitrary pair of real numbers x and y there exists a unique point A in the xy-plane for which x will be its abscissa and y its ordinate.

Indeed, suppose for definiteness $x > 0$ and $y < 0$. Let us take a point A_x at the distance x to the right the origin O on the x-axis and a point A_y at the distance y from the origin below the x-axis. We then draw through the points A_x and A_y straight lines parallel to the y- and x-axes respectively (Fig. 4). These lines will intersect at a point A whose abscissa is obviously x, and ordinate is y. In other cases $x < 0$, $y > 0$, $x > 0$, $y > 0$ and $x < 0$, $y < 0$. The proof is analogous.

2. Distance Between Two Points

Let there be given two points on the xy-plane: A_1 has the coordinates x_1, y_1 and A_2 has the coordinates x_2, y_2. *It is required to express the distance between the points A_1 and A_2 in terms of their coordinates.*

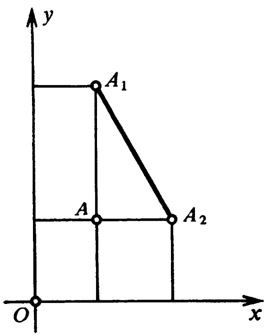

Fig. 5

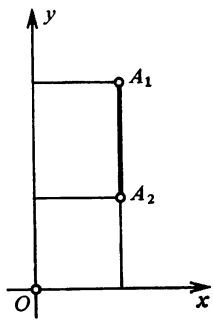

Fig. 6

Suppose $x_1 \neq x_2$ and $y_1 \neq y_2$. Through points A_1 and A_2 we draw straight lines parallel to the coordinate axes (Fig. 5). The distance between the points A and A_1 is equal to $|y_1 - y_2|$, and the distance between the points A and A_2 is equal to $|x_1 - x_2|$. Applying the

Pythagoras' theorem to the right-angled triangle A_1AA_2, we get

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = d^2, \qquad (*)$$

where d is the distance between the points A_1 and A_2.

Though the formula (*) for determining the distance between points has been derived by us proceeding from the assumption that $x_1 \neq x_2$, $y_1 \neq y_2$, it is true for other cases as well. Indeed, for $x_1 = x_2$, $y_1 \neq y_2$ d is equal to $|y_1 - y_2|$ (Fig. 6). The formula (*) yields the same result. For $x_1 \neq x_2$, $y_1 = y_2$ we get a similar result. If $x_1 = x_2$, $y_1 = y_2$ the points A_1 and A_2 coincide and the formula (*) yields $d = 0$.

As an exercise, *let us find the coordinates of the centre of a circle circumscribed about a triangle with the vertices* (2, —2), (—2, 2) and (1, 5).

Let (x, y) be the centre of the circumcircle. It is equidistant from the vertices of the triangle. Equating the squares of the distances we derive the following equations for the coordinates x and y. Thus, we have

$$(x - 2)^2 + (y + 2)^2 = (x + 2)^2 + (y - 2)^2,$$

$$(x - 2)^2 + (y + 2)^2 = (x - 1)^2 + (y - 5)^2.$$

After obvious transformations, we obtain

$$-x + y = 0, \quad -x + 7y - 9 = 0.$$

And hence, we obtain

$$x = \frac{3}{2} \text{ and } y = \frac{3}{2}.$$

3. Dividing a Line Segment in a Given Ratio

Let there be given two different points on the xy-plane: $A_1(x_1, y_1)$ and $A_2(x_2, y_2)$. *Find the coordinates x and y of the point A which divides the line segment A_1A_2 in the ratio $\lambda_1 : \lambda_2$.*

Suppose the segment A_1A_2 is not parallel to the x-axis. Projecting the points A_1, A, A_2 on the y-axis, we have (Fig. 7)

$$\frac{A_1A}{AA_2} = \frac{\bar{A}_1\bar{A}}{\bar{A}\bar{A}_2} = \frac{\lambda_1}{\lambda_2}.$$

Since the points $\bar{A}_1$, $\bar{A}_2$, $\bar{A}$ have the same ordinates as the points A_1, A_2, A, respectively, we get

$$\bar{A}_1\bar{A} = |y_1 - y|, \quad \bar{A}\bar{A}_2 = |y - y_2|.$$

Consequently,

$$\frac{|y_1 - y|}{|y - y_2|} = \frac{\lambda_1}{\lambda_2}.$$

Since $\bar{A}$ lies between $\bar{A}_1$ and $\bar{A}_2$, $y_1 - y$ and $y - y_2$ have the same sign. Therefore

$$\frac{|y_1-y|}{|y-y_2|}=\frac{y_1-y}{y-y_2}=\frac{\lambda_1}{\lambda_2}.$$

Whence we find

$$y=\frac{\lambda_2 y_1+\lambda_1 y_2}{\lambda_1+\lambda_2}. \qquad (*)$$

If the line segment A_1A_2 is parallel to the x-axis, then

$$y_1 = y_2 = y.$$

The same result is obtained from the formula (*) which is thus true for any locations of the points A_1 and A_2.

The abscissa of the point A is found analogously. For the abscissa we get the formula

$$x=\frac{\lambda_2 x_1+\lambda_1 x_2}{\lambda_1+\lambda_2}.$$

As an exercise, let us prove Ceva's theorem from elementary geometry. It states: *If the sides of a triangle are divided by the concur-*

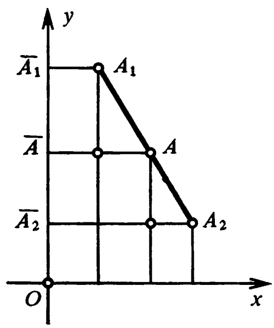

Fig. 7

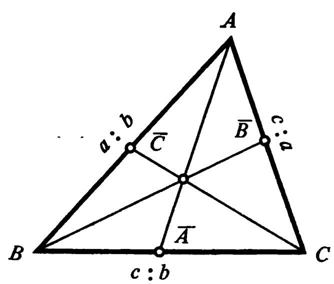

Fig. 8

rent cevians (cevian of a triangle is a line segment that joins the vertex of the triangle to a point on the opposite side), then these cevians are concurrent in the ratio $a : b$, $c : a$, $b : c$.

Let $A\ (x_1, y_1)$, $B\ (x_2, y_2)$, and $C\ (x_3, y_3)$ be the vertices of the triangle and $\bar{A}$, $\bar{B}$, $\bar{C}$ the points of division of the opposite sides (Fig. 8). The coordinates of the point $\bar{A}$ are:

$$x=\frac{bx_2+cx_3}{b+c}, \qquad y=\frac{by_2+cy_3}{b+c}.$$

Let us divide the line segment $\bar{A}A$ in the ratio $(b + c) : a$. Then the coordinates of the point of division will be

$$x = \frac{ax_1 + bx_2 + cx_3}{a+b+c},$$

$$y = \frac{ay_1 + by_2 + cy_3}{a+b+c}.$$

If the segment $\bar{B}B$ is divided in the ratio $(a + c) : b$, then we get the same coordinates of the point of division. The same coordinates

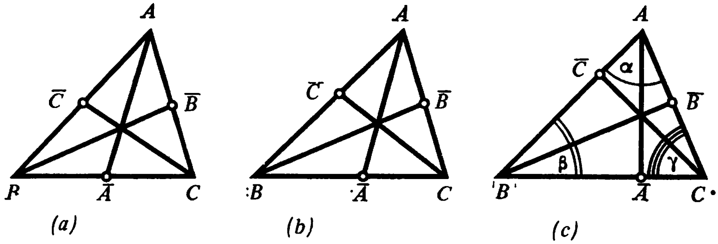

Fig. 9

are obtained when dividing the segment $\bar{C}C$ in the ratio $(a + b) : c$. Hence, the segments $\bar{A}A$, $\bar{B}B$, and $\bar{C}C$ have a point in common, which was required to be proved.

We should note that theorems from the course of elementary geometry on intersection of medians, bisectors, and altitudes in the triangle are the particular cases of Ceva's theorem. We now proceed to clarify this.

For medians (Fig. 9*a*) $A\bar{C} : \bar{C}B = 1 : 1$, $B\bar{A} : \bar{A}C = 1 : 1$, $C\bar{B} : \bar{B}A = 1 : 1$. For bisectors (Fig. 9*b*) $A\bar{C} : \bar{C}B = AC : BC$, $B\bar{A} : \bar{A}C = AB : AC$, $C\bar{B} : \bar{B}A = BC : AB$. For altitudes (Fig. 9*c*) $A\bar{C} : \bar{C}B = \frac{C\bar{C}}{\tan\alpha} : \frac{C\bar{C}}{\tan\beta} = \frac{1}{\tan\alpha} : \frac{1}{\tan\beta}$, $B\bar{A} : \bar{A}C = \frac{1}{\tan\beta} : \frac{1}{\tan\gamma}$, $C\bar{B} : \bar{B}A = \frac{1}{\tan\gamma} : \frac{1}{\tan\alpha}$. We see that in all cases the conditions of Ceva's theorem are satisfied.

4. Equation of a Curve. Equation of a Circle

Let there be given a curve on the xy-plane (Fig. 10). The equation $\varphi(x, y) = 0$ is called the *equation of a curve in implicit form* if the coordinates x, y of any point of this curve satisfy the equation and any pair of real numbers x, y, satisfying the equation $\varphi(x, y) = 0$ represents the coordinates of the point on the curve. A curve is obviously defined by its particular equation, therefore we may speak of assigning a curve by its equation.

In analytic geometry two problems are often considered: (1) given the geometric properties of a curve, form its equation; (2) given the equation of a curve, define its geometric properties. Let us consider these problems as applied to the circle which is a simple curve.

Suppose A_0 (x_0, y_0) is an arbitrary point in the xy-plane, and R is any positive number. Let us form the equation of a circle with centre A_0 and radius R (Fig. 11).

Let A (x, y) be an arbitrary point of the circle. Its distance from the centre A_0 is R. The square of the distance of the point A from A_0

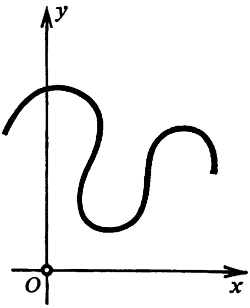

Fig. 10

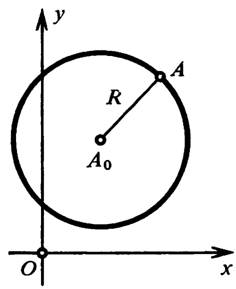

Fig. 11

is equal to $(x - x_0)^2 + (y - y_0)^2$. Thus, the coordinates x, y of any point A of the circle satisfy the equation

$$(x - x_0)^2 + (y - y_0)^2 - R^2 = 0. \qquad (*)$$

Conversely, any point A whose coordinates satisfy the equation (*) belongs to the circle, since its distance from A_0 is equal to R.

In conformity with the above definition, the equation (*) *is the equation of a circle with centre* A_0 *and radius* R.

We now consider another problem for the curve given by the equation

$$x^2 + y^2 + 2ax + 2by + c = 0 \quad (a^2 + b^2 - c > 0).$$

This equation can be rewritten in the following equivalent form:

$$(x+a)^2 + (y+b)^2 - \left(\sqrt{a^2+b^2-c}\right)^2 = 0.$$

From this equation we can see that any point (x, y) of the curve is at the distance of $\sqrt{a^2 + b^2 - c}$ from the point $(-a, -b)$, and, hence, the *curve is a circle with centre* $(-a, -b)$ *and radius* $\sqrt{a^2 + b^2 - c}$.

Let us consider the following problem as an example illustrating the application of the method of analytic geometry: *Find the locus* *of points in a plane the ratio of whose distances from two given points A and B is constant and is equal to* $k \neq 1$. (The *locus* is a set of all points which possess the given geometric property. In the case under consideration we speak of such points in the plane for which the ratio of distances from the two given points A and B is constant.)

Suppose that $2a$ is the distance between the points A and B. We introduce a rectangular Cartesian coordinate system in the plane taking the straight line AB as the x-axis and the midpoint of the segment AB for the origin. Let, for definiteness, the point A lie on the positive x-axis. The coordinates of the point A will then be: $x = a$, $y = 0$, and the coordinates of the point B will be: $x = -a$, $y = 0$. Let (x, y) be an arbitrary point of the locus. The squares of its distances from the points A and B are respectively equal to $(x - a)^2 + y^2$ and $(x + a)^2 + y^2$. The equation of the locus is

$$\frac{(x-a)^2+y^2}{(x+a)^2+y^2} = k^2,$$

or

$$x^2+y^2+\frac{2(k^2+1)}{k^2-1}ax+a^2=0.$$

Thus the locus is a *circle* (Apollonius' circle).

5. Parametric Equations of a Curve

Assume that a point A moves along a curve, and by the time t its coordinates are: $x = \varphi(t)$ and $y = \psi(t)$. Simultaneous equations

$$x = \varphi(t), \quad y = \psi(t),$$

which specify the coordinates of an arbitrary point on the curve as functions of the parameter t are called *parametric equations.*

The parameter t need not be time, it may be any other quantity which describes the position of a point on a curve.

Let us now form a parametric equation for a circle.

Suppose we have a circle with centre at the origin, and of radius R. The position of a point A on the circle can be described by the angle α formed by the radius OA and by the x-axis (Fig. 12). The coordinates of the point A are $R\cos\alpha$, $R\sin\alpha$, and, consequently, the equation of the circle has a form:

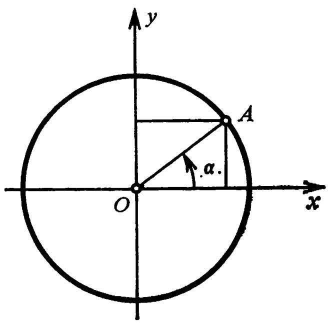

Fig. 12

$$x = R\cos\alpha, \quad y = R\sin\alpha.$$

From a parametric equation of a curve

$$x = \varphi(t), \quad y = \psi(t), \tag{*}$$

we can obtain its equation in implicit form:

$$f(x, y) = 0.$$

To do this it is sufficient to eliminate the parameter t from the equations (*), by having found it from one equation and substituting it into the other, or using any other method.

For instance, to get the equation of a circle specified by the parametric equations (i.e. implicitly) it is sufficient to square both equalities and add them termwise. We then obtain the equation $x^2 + y^2 = R^2$.

The elimination of the parameter from the parametric equation of a curve not always yields an equation in implicit form in the sense of the above definition. The points not belonging to the curve may satisfy this equation. Now let us consider two examples.

Assume that a curve γ is given by the parametric equations

$$x = a\cos t, \quad y = b\sin t, \quad 0 \leqslant t < 2\pi.$$

Dividing these equations by a and b, respectively, squaring and adding them termwise, we get the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This equation is obviously satisfied by all the points belonging to the curve γ. Conversely, if the point (x, y) satisfies this equation, then there can be found an angle t for which $x/a = \cos t$, $y/b = \sin t$, and, consequently, any point in the plane which satisfies this equation, belongs to the curve γ.

Let now a curve γ be represented by the following equations

$$x = a\cosh t, \quad y = b\sinh t, \quad -\infty < t < +\infty,$$

where

$$\cosh t = \frac{1}{2}(e^t + e^{-t}), \text{ and } \sinh t = \frac{1}{2}(e^t - e^{-t}).$$

Dividing these equations by a and b, respectively, and then squaring them and subtracting termwise, we get the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The points of the curve γ satisfy this equation. But not any point which satisfies the equation belongs to γ, for instance, the point $(-a, 0)$. It satisfies the equation, but does not belong to the curve, since on the curve γ $a\cosh t \neq -a$.

Sometimes the equation of a curve represented in implicit form is understood in a wider sense, viz., all the points satisfying the equation need not belong to the curve.

6. Points of Intersection of Curves

Let there be given two curves in the xy-plane: the curve γ_1 defined by the equation

$$f_1(x, y) = 0,$$

and the curve γ_2 specified by the equation

$$f_2(x, y) = 0.$$

We now find the points of intersection of the curves γ_1 and γ_2, i.e. the coordinates of these points.

Let $A(x, y)$ be the point of intersection of the curves γ_1 and γ_2. As the point A lies on the curve γ_1, its coordinates satisfy the equation $f_1(x, y) = 0$. Also, as the point A lies on the curve γ_2, its coordinates satisfy the equation $f_2(x, y) = 0$. Thus, the coordinates of any point of intersection of the curves γ_1 and γ_2 satisfy simultaneous equations

$$f_1(x, y) = 0, \quad f_2(x, y) = 0.$$

Conversely, any real solution to this system of equations yields the coordinates of one of the points of intersection of the curves.

If the curve γ_1 is given by the equation

$$f_1(x, y) = 0,$$

and the curve γ_2 is given by the parametric equations

$$x = \varphi(t), \quad y = \psi(t),$$

then the coordinates x, y of the points of intersection satisfy a system of three simultaneous equations

$$f_1(x, y) = 0, \quad x = \varphi(t), \quad y = \psi(t).$$

If both curves are parametric

$$\gamma_1 : x = \varphi_1(t), \quad y = \psi_1(t);$$
$$\gamma_2 : x = \varphi_2(\tau), \quad y = \psi_2(\tau),$$

then the coordinates x, y of the points of intersection satisfy the following system of four simultaneous equations:

$$x = \varphi_1(t), \quad y = \psi_1(t),$$
$$x = \varphi_2(\tau), \quad y = \psi_2(\tau).$$

Example. Find the points of intersection of the circles

$$x^2 + y^2 = 2ax, \quad x^2 + y^2 = 2by.$$

Subtracting the equations termwise, we find $ax = by$. Substituting $y = ax/b$ in the first equation, we get

$$\left(1 + \frac{a^2}{b^2}\right) x^2 - 2ax = 0.$$

Whence

$$x_1 = 0, \quad x_2 = \frac{2ab^2}{a^2+b^2},$$

the corresponding ordinates being

$$y_1 = 0, \quad y_2 = \frac{2ba^2}{a^2+b^2}.$$

The required points of intersection are (0, 0) and $\left(\frac{2ab^2}{a^2+b^2}, \frac{2ba^2}{a^2+b^2}\right)$.

7. Relative Position of Two Circles

Consider two circles of radii a and b, respectively, the centre-to-centre distance being c. What is their mutual position?

Let O and O_1 be the centres of the circles. We take O to be the origin of coordinates and the half-line OO_1 the x-axis to the right of the origin. The equations of the circles are

$$x^2 + y^2 = a^2, \quad (x - c)^2 + y^2 = b^2. \tag{*}$$

If the circles intersect, the coordinates x, y of the point of intersection obey both equations (*). Conversely, if the system of equations (*) has a solution, i.e. there exist x and y such that they meet both equations, then they are the coordinates of the point of intersection of the circles. The number of points of intersection (if any) equals the number of the solutions of the system.

Let us now solve the system (*). To this end, we first subtract the equations term by term. We get $2cx - c^2 = a^2 - b^2$. Hence $x = (a^2 + c^2 - b^2)/2c$. Substituting this value of x into the first equation gives

$$\left(\frac{a^2+c^2-b^2}{2c}\right) + y^2 = a^2.$$

Hence

$$y = \pm\sqrt{a^2 - \left(\frac{a^2+c^2-b^2}{2c}\right)^2}.$$

We transform the expression under the radical as the difference of squares

$$\left(a+\frac{a^2+c^2-b^2}{2c}\right)\left(a-\frac{a^2+c^2-b^2}{2c}\right)$$
$$=\frac{1}{4c^2}(2ac+a^2+c^2-b^2)(2ac-a^2-c^2+b^2)$$
$$=\frac{1}{4c^2}[(a+c)^2-b^2][b^2-(a-c)^2]$$
$$=\frac{1}{4c^2}(a+b+c)(a+c-b)(b+a-c)(b-a+c).$$

Thus

$$y=\pm\frac{1}{2c}\sqrt{(a+b+c)(a+c-b)(a+b-c)(b+c-a)}.$$

It follows that if $a+c>b$, $a+b>c$ and $b+c>a$, then the radicand is positive, and hence the system (*) has solutions, of which there are two: one of them has the root with a plus sign, and the other with a minus sign. And so the circles intersect at two points.

If at least one of the factors $a+c-b$, $a+b-c$, $b+c-a$ is zero, then the system (*) has one solution, i.e. the circles touch each other.

If one of the factors in the radicand is negative, then the system (*) has no solutions and the circles do not intersect. Two factors in the radicand cannot be negative, since then their sum would be negative. But it is known to be positive. For example, if $a+c-b<0$ and $a+b-c<0$, then $(a+c-b)+(a+b-c)=2a<0$, which is impossible. The situation will be the same in other cases.

Consequently, *if one of the numbers a, b, c is larger than the sum of the two others, then the circles do not intersect*; *if one of them equals the sum of the two others, then the circles touch*; *if one of them is less than the sum of the two others, then the circles intersect at two points.*

This examination enables us to solve the issue of the existence of a triangle with given sides. So for the given line segments of lengths a, b and c to be the sides of some triangle it is sufficient for the largest of a, b, or c to be less than the sum of the two others. Really, let us take a segment AB of length c and draw circles with centres A and B, of radii a and b, respectively. As proved above, these circles intersect at a certain point C. The triangle ABC has the sides $AB = c$, $AC = a$, and $BC = b$.

EXERCISES TO CHAPTER I

1. Given two points on a straight line parallel to the x-axis. The ordinate of one of them is $y = 2$. Find the ordinate of the other point.

2. From a point A (2, 3) a perpendicular is dropped on the x-axis. Find the coordinates of the foot of the perpendicular.

3. Through a point A (2, 3) a straight line parallel to the x-axis is drawn. Find the coordinates of the point where it cuts the y-axis.

4. Find the locus of points in the xy-plane with the abscissa $x = 3$.

5. Given points A (—3, 2), B (4, 1), show that the line segment AB cuts the y-axis, but not the x-axis.

6. What part of the y-axes (above or below the x-axis) is cut by the segment AB in the previous problem?

7. Find the distance from the point (—3, 4) to the x-axis (y-axis).

8. Consider a point with ordinate $y = 2$ lying on the bisector of the first quadrant. Find the abscissa of the point.

9. Solve the previous problem, given that the point lies on the bisector of the second quadrant.

10. Find the locus of points in the xy-plane, with $x = y$.

11. Find the locus in the xy-plane with $x = -y$.

12. What is the position of the points of the xy-plane for which (a) $|x| = a$, (b) $|x| = |y|$?

13. What is the position of the points of the xy-plane for which (a) $|x| < a$, (b) $|x| < a$, $|y| < b$?

14. Find the coordinates of a point symmetric to the point A (x, y) about the x-axis (y-axis, the origin).

15. Find the coordinates of a point symmetric to the point A (x, y) about the bisector of the first (second) quadrant.

16. How will the coordinates of the point A (x, y) change if the y-axis is taken as the x-axis, and vice versa?

17. Given points A (4, —2), B (1, 2), C (—2, 6), find the distances between these points, taken in pairs.

18. Show that points A, B, C in the previous problem lie on the same straight line. Which of the points lies in between?

19. Find on the x-axis a point equidistant from points (1, 2) and (2, 3).

20. Find a point equidistant from the coordinate axes and from the point (3, 6).

21. Given the coordinates of two vertices A and B of an equilateral triangle ABC, find the coordinates of the third vertex. Consider the case A (0, 1), B (2, 0).

22. Given the coordinates of two adjacent vertices A and B of a square $ABCD$, find the coordinates of the remaining vertices. Consider the case A (1, 0), B (0, 1).

23. What condition must the coordinates of the vertices of a triangle ABC satisfy for a triangle to have a right angle at the vertex C?

24. What condition must the coordinates of the vertices of a triangle ABC satisfy for the angle A to be larger than the angle B?

25. A quadrilateral $ABCD$ is specified by the coordinates of its vertices. Determine whether or not it is inscribed in a circle.

26. Prove that for any real a, a_1, a_2, b, b_1, b_2 there exists the following inequality

$$\sqrt{(a_1-a)^2+(b_1-b)^2}+\sqrt{(a_2-a)^2+(b_2-b)^2} \geqslant \sqrt{(a_1-a_2)^2+(b_1-b_2)^2}.$$

To what geometric fact does it correspond?

27. Given the three vertices of a parallelogram $ABCD$: A (1, 0), B (2, 3), C (3, 2), find the coordinates of the fourth vertex and of the point of intersection of the diagonals O.

28. Show that points A (—1, —2), B (2, —5), C (1, —2), D (—2, 1) are the vertices of a parallelogram. Find the point of intersection of its diagonals.

29. Given one end of a line segment (1, 1) and its midpoint (2, 2), find the second end.

30. Show that points (3, 0), (1, 0), (1, —2), (3, —2) are the vertices of a square.

31. Given the coordinates of the vertices of a triangle: (x_1, y_1), (x_2, y_2), and (x_3, y_3), find the coordinates of the point of intersection of the medians.

32. Given the coordinates of midpoints of the sides of a triangle (x_1, y_1), (x_2, y_2), and (x_3, y_3), find the coordinates of its vertices.

33. Given a triangle with the vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3), find the coordinates of vertices of a similar triangle and of a similarly located triangle with the ratio of similitude λ and the centre of similitude at point (x_0, y_0).

34. The point A is said *to divide* the line segment A_1A_2 *externally* in the ratio $\lambda_1 : \lambda_2$ if this point lies on a straight line joining the points A_1 and A_2 outside the segment A_1A_2 and the ratio of its distances from the points A_1 and A_2 is $\lambda_1 : \lambda_2$. Show that the coordinates of the point A are expressed in terms of the coordinates (x_1, y_1), (x_2, y_2) of the points A_1 and A_2 by the formulas

$$x=\frac{\lambda_2 x_1-\lambda_1 x_2}{\lambda_2-\lambda_1}, \quad y=\frac{\lambda_2 y_1-\lambda_1 y_2}{\lambda_2-\lambda_1}.$$

35. Two line segments are specified by the coordinates of their end-points. How can we find out, without using a drawing, whether the segments intersect or not?

36. *The centre of gravity of two masses* μ_1 and μ_2 located at points A_1 (x_1, y_1) and A_2 (x_2, y_2) is defined as a point A which divides the segment A_1A_2 in the ratio $\mu_2 : \mu_1$. Thus, its coordinates are:

$$x=\frac{\mu_1 x_1+\mu_2 x_2}{\mu_1+\mu_2}, \quad y=\frac{\mu_1 y_1+\mu_2 y_2}{\mu_1+\mu_2}.$$

The centre of gravity of n masses μ_i situated at points A_i is determined by induction. Indeed, if A_n' is the centre of gravity of the first n — 1 masses, then the centre of gravity of all n masess is determined as the centre of gravity of two masses: μ_n located at point A_n, and $\mu_1 + \mu_2 + \ldots + \mu_{n-1}$, situated at point A'_n. Derive the formulas for the coordinates of the centre of gravity of the masses μ_i situated at points A_i (x_i, y_i):

$$x = \frac{\mu_1 x_1 + \ldots + \mu_n x_n}{\mu_1 + \ldots + \mu_n}, \qquad y = \frac{\mu_1 y_1 + \ldots + \mu_n y_n}{\mu_1 + \ldots + \mu_n}.$$

37. Find the centre of a circle lying on the x-axis, given that the circle passes through a point (1, 4) and its radius is 5.

38. What are specific features in the position of the circle

$$x^2 + y^2 + 2ax + 2by + c = 0 \quad (a^2 + b^2 - c > 0)$$

with respect to the coordinate system if (a) $a = 0$; (b) $b = 0$; (c) $c = 0$; (d) $a = 0$, $b = 0$; (e) $a = 0$, $c = 0$; (f) $b = 0$, $c = 0$?

39. Show that if we substitute the coordinates of any point lying outside the circle in the left-hand side of the equation of a circle then we shall obtain the square of the length of a tangent drawn from this point to the circle.

40. The *power of a point A* with reference to a circle is defined as the product of the segments of a secant drawn through the point A taken with the plus sign for points external to the circle and with the minus sign for points internal to the circle. Show that the left-hand member of the equation of a circle $x^2 + y^2 + 2ax + 2by + c = 0$ gives the power of this point with reference to a circle when the coordinates of an arbitrary point are substituted in it.

41. Form the equation for the locus of points of the xy-plane the sum of whose distances from the two given points F_1 $(c, 0)$ and F_2 $(-c, 0)$ is constant and equal to $2a$ (an *ellipse*). Show that the equation is reduced to the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b^2 = a^2 - c^2$.

42. Form the equation for the locus of points of the xy-plane the difference of whose distances from the two given points F_1 $(c, 0)$ and F_2 $(-c, 0)$ is constant and is equal to $2a$ (a *hyperbola*). Show that the equation is reduced to the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = c^2 - a^2$.

43. Write the equation for the locus of points of the xy-plane which are equidistant from the point F $(0, p)$ and the x-axis (a *parabola*).

44. Show that the following parametric equations

$$x = R\cos t + a, \quad y = R\sin t + b$$

represent a circle of radius R with centre at the point (a, b).

45. Form the equation for a curve described by the point on the line segment of length a when the end-points of the segment move along the coordinate axes. (The segment is divided by this point in the ratio $\lambda : \mu$.) Take as the parameter the angle made by the segment with the x-axis. What is the shape of the curve if $\lambda : \mu = 1$?

46. A triangle slides along the coordinate axes with two of its vertices. Write the equation for the curve described by the third vertex (Fig. 13).

47. Form the equation for a curve described by a point A on a circle of radius R which rolls along the x-axis (Fig. 14). Take as the parameter the path s covered by the centre of the circle. Assume that at the initial moment ($s = 0$) the point A coincides with the origin.

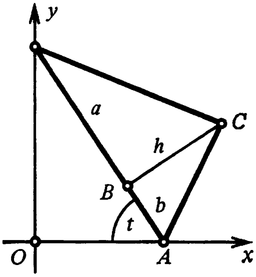

Fig. 13

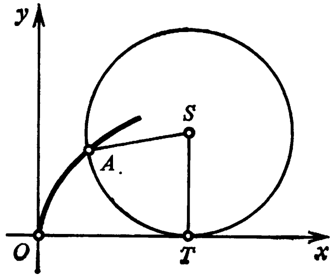

Fig. 14

48. A curve is given by the equation

$$ax^2 + bxy + cy^2 + dx + ey = 0.$$

Show that, by introducing the parameter $t = y/x$, we can obtain the following equations for the parametric curve:

$$x = -\frac{d+et}{a+bt+ct^2}, \quad y = -\frac{dt+et^2}{a+bt+ct^2}.$$

49. Form the equation of the circle with centre at a point (1, 2), given that it touches the x-axis.

50. Form the equation of the circle, centre (—3, 4), given that it passes through the origin of coordinates.

51. Show that the circle $x^2 + y^2 + 2ax + 1 = 0$ does not meet the y-axis.

52. Show that the circle $x^2 + y^2 + 2ax = 0$ touches the y-axis.

53. What condition must the coefficients of the equation of a circle

$$x^2 + y^2 + 2ax + 2by + c = 0$$

satisfy for the circle (a) not to intersect the x-axis; (b) to intersect the x-axis at two points, (c) to touch the x-axis?

54. What condition must the coefficients of the following equations of circles

$$x^2 + y^2 + 2a_1x + 2b_1y + c_1 = 0,$$
$$x^2 + y^2 + 2a_2x + 2b_2y + c_2 = 0,$$

satisfy for the circles (a) to intersect; (b) to touch?

55. Find the points of intersection of the two circles:

$$x^2 + y^2 = 1 \text{ and } x = \cos t + 1, \quad y = \sin t.$$

56. Find the points of intersection of the two parametric curves

$$x = s^2 + 1, \qquad x = t^2,$$

and

$$y = s, \qquad y = t + 1.$$

57. Show that the points of intersection of the curves

$$ax^2 + by^2 = c, \qquad Ax^6 + By^6 = C$$

are symmetric about the coordinate axes.

Chapter II

VECTORS IN THE PLANE

1. Translation

We introduce Cartesian coordinates x, y on the plane. A transformation of a figure F, under which its points (x, y) are carried into $(x + a, y + b)$, where a and b are two constants, is called a *translation* (Fig. 15). A translation is given by the formulas

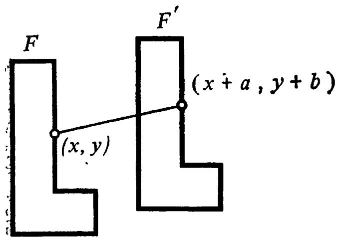

Fig. 15

$$x' = x + a, \quad y' = y + b \quad (*)$$

expressing the coordinates x', y' of the point into which (x, y) is carried.

A translation is a motion. In fact, two arbitrary points A (x_1, y_1) and B (x_2, y_2) are transformed into A' $(x_1 + a, y_1 + b)$, B' $(x_2 + a, y_2 + b)$ and $AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$, $A'B'^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$. Hence, $AB = A'B'$. Thus, the transformation is distance-preserving and, therefore, a motion.

The name "translation" is justified by *the shift of points along parallel or coincident straight lines through the same distance*. Indeed, let two points A (x_1, y_1) and B (x_2, y_2) be sent into A' $(x_1 + a, y_1 + b)$, B' $(x_2 + a, y_2 + b)$ (Fig. 16). The midpoint of the line segment AB' as well as that of $A'B$ have coordinates

$$x = \frac{x_1 + x_2 + a}{2}, \qquad y = \frac{y_1 + y_2 + b}{2}.$$

Hence, the diagonals of the quadrilateral $AA'B'B$ meet and bisect each other. It is therefore a parallelogram whose opposite sides AA' and BB' are parallel and equal.

Note that the other two sides AB and $A'B'$ are also parallel. Consequently, *under a translation, a straight line is transformed into a parallel line (or into itself).*

For any two points A and A', there is one, and only one, translation under which A is carried into A'.

Proof. We start with uniqueness. Let X be an arbitrary point of the figure, and sent by the translation into X' (Fig. 17). As we know,

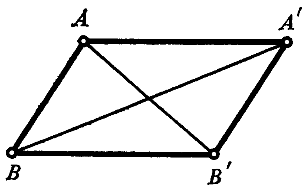

Fig. 16

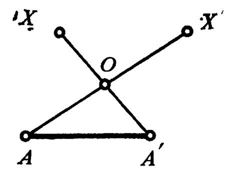

Fig. 17

the line segments XA' and AX' possess a common midpoint O. The specification of X determines O uniquely, whereas A and O uniquely determine X', since O is the midpoint of AX'. The uniqueness in the determination of X' is just what implies that of the translation.

To prove the existence of a translation sending A into A', we introduce Cartesian coordinates on the plane. Let a_1, a_2 be the coordinates of A, and a_1', a_2' those of A'. The translation given by

$$x' = x + a_1' - a_1, \quad y' = y + a \quad - a_2$$

sends A into A'. In fact, when $x = a_1$, and $y = a_2$, we obtain $x' = a_1'$, $y' = a_2'$. Q.E.D.

It follows from the uniqueness of a translation sending a given point A into a given point A', which is established without involving a coordinate system, that *a translation is given by formulas of the form*

$$x' = x + a, \quad y' = y + b$$

in any Cartesian system of coordinates. The constants a and b, certainly, depend on the choice of a coordinate system.

As an exercise, we solve the following.

Under a translation, the point (1, 1) *is sent into* (—1, 0). *What point will the origin be transformed into*?

Solution. Any translation is given by the formulas $x' = x + a$, $y' = y + b$. Since (1, 1) is carried into (—1, 0), we have $-1 = 1 + a$, $0 = 1 + b$. Hence, $a = -2$, $b = -1$. Thus, our translation is given by $x' = x - 2$, $y' = y - 1$. Substituting into the formulas the coordinates of the origin, $x = 0$, $y = 0$, we obtain $x' = -2$, $y' = -1$. Thus, the origin is transformed into the point $(-2, -1)$.

A transformation inverse to a translation is a translation. Two translations performed one after another again yield a translation.

Proof. Any translation is given by formulas of the form

$$x' = x + a, \quad y' = y + b.$$

The inverse transformation is determined by formulas of the same form

$$x = x' - a, \quad y = y' - b,$$

and is, therefore, a translation, thus proving the first statement.

Now, consider two translations specified by

$$x' = x + a, \quad y' = y + b,$$
$$x'' = x' + c, \quad y'' = y' + d.$$

A transformation obtained on performing these two consecutively is

$$x'' = x + a + c, \quad y'' = y + b + d$$

which is a translation, and the theorem is proved completely.

2. Modulus and the Direction of a Vector

We will call a directed line segment a *vector* (Fig. 18), and denote it by a small letter **a**, **b**, **c**, . . . in bold type. Sometimes, a vector is given by specifying the ends of a line segment representing it. For instance, the vector in Fig. 18 can be denoted by $\overrightarrow{AB}$. With this method of denoting a vector **a**, the point A is called its *origin*, and B its *end-point*. If we denote a vector by means of the ends of a line segment representing it, then we always place the origin first. Sometimes, the notation $\bar{a}$ or $\vec{a}$ is used (to be read "a vector **a**").

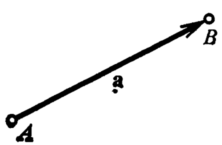

Fig. 18

Two half-lines are said to be *co-directional* if they can be made to coincide under a translation, i.e., if there exists a translation which would transform one of them into the other.

If two half-lines a and b are co-directional, and two half-lines b and c are also co-directional, then a and c are co-directional. Indeed, since a and b are co-directional, there exists a translation which transforms a

into b. Since b and c are co-directional, there exists a translation which transforms b into c. Carried out one after the other, these two yield a translation to transform a into c. Hence, a and c are co-directional.

Two half-lines are said to be *opposite* if each of them is co-directional with one half-line complementary to the other.

Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are said to be *co-directional* if the half-lines AB and CD are co-directional. The *modulus* (or *magnitude*) of a vector is the length of a line segment representing it. The modulus of a vector **a** is denoted by $|\mathbf{a}|$, and that of $\overrightarrow{AB}$ by AB.

Two vectors are said to be *equal* if they can be made to coincide under a translation, which means that there exists a translation transforming the origin and end-point of one into those of the other, respectively. Hence, *equal vectors are co-directional and equal in modulus. Conversely, if vectors are co-directional and equal in modulus, then they are equal.* In fact, let $\overrightarrow{AB}$ and $\overrightarrow{CD}$ be two co-directional vectors equal in modulus. The translation transforming C into A makes the half-line CD coincident with the half-line AB, since they are co-directional. And because the line segments AB and CD are equal, the point D is then made coincident with the point B, i.e., the translation sends the vector $\overrightarrow{CD}$ into the vector $\overrightarrow{AB}$. Therefore, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal.

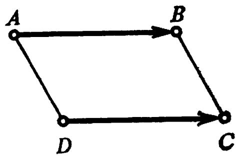

Fig. 19

Given a parallelogram $ABCD$, prove that the vectors $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are equal.

Solution. Let $\overrightarrow{AB}$ be subjected to a translation sending the point A into D (Fig. 19). A is then shifted along the straight line AD, and therefore, B along the parallel straight line BC. The straight line AB is transformed into a parallel line, and, consequently, into the straight line DC. Therefore, B is sent into the point C. Thus, our translation transforms $\overrightarrow{AB}$ into $\overrightarrow{DC}$; hence, they are equal.

Denoting a vector by its end-points ($\overrightarrow{AB}$), it is natural, and, as we see below, expedient, to consider the vector whose end-points coincide ($\overrightarrow{AA}$). Call it the zero vector, and denote it by **0**. Its direction is not spoken of, and the modulus is assumed to be zero. All zero vectors are equal by definition.

It follows from the translation properties that *one, and only one, vector equal to a given vector can be marked off from any point.* For proof, it suffices to carry out a translation under which the origin of the vector will be transformed into the given point.

3. Components of a Vector

Let a vector **a** have a point $A_1(x_1, y_1)$ as the origin, and a point $A_2(x_2, y_2)$ as the end-point. Its *components* are the values $a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$. We write **a** (a_1, a_2), or simply $\overrightarrow{(a_1 a_2)}$. The zero vector components are zeros.

It follows from the formula expressing the distance between two points in terms of their coordinates that the modulus of a vector with components a_1, a_2 is equal to $\sqrt{a_1^2 + a_2^2}$.

Equal vectors have equal respective components. Consequently, if the respective components of two vectors are equal, then the vectors are also equal.

Proof. Let $A_1(x_1, y_1)$ and $A_2(x_2, y_2)$ be the origin and end-point of a vector **a**. Since an equal vector **a**′ is obtained from **a** by a translation, then its origin and end-point are $A_1'(x_1 + c, y_1 + d)$ and $A_2'(x_2 + c, y_2 + d)$, respectively. Hence, both **a** and **a**′ have the same components $x_2 - x_1$, $y_2 - y_1$.

To prove the converse statement, we suppose that the corresponding components of the vectors $\overrightarrow{A_1A_2}$ and $\overrightarrow{A_1'A_2'}$ are equal, and show that the vectors themselves are equal. Let x_1' and y_1' be the coordinates of the point A_1', and x_2', y_2' those of the point A_2'. Given that $x_2 - x_1 = x_2' - x_1'$, $y_2 - y_1 = y_2' - y_1'$, we have $x_2' = x_2 + x_1' - x_1$, $y_2' = y_2 + y_1' - y_1$, and the translation specified by the formulas $x' = x + x_1' - x_1$, $y' = y + y_1' - y_1$, sends A_1 to A_1', and A_2 to A_2', i.e., $\overrightarrow{A_1A_2}$ and $\overrightarrow{A_1'A_2'}$ are equal. Q.E.D.

Given three points $A(1, 1)$, $B(-1, 0)$, $C(0, 1)$, *find a point* $D(x, y)$ *so that the vectors* $\overrightarrow{AB}$ *and* $\overrightarrow{CD}$ *are equal.*

Solution. The components of $\overrightarrow{AB}$ are $(-2, -1)$, whereas those of $\overrightarrow{CD}$ $(x - 0, y - 1)$. Since $\overrightarrow{AB} = \overrightarrow{CD}$, $x - 0 = -2$, $y - 1 = -1$. Hence, the coordinates of the point D are $x = -2$, $y = 0$.

4. Addition of Vectors

The *sum of two vectors* **a** and **b** with components a_1, a_2 and b_1, b_2 is a vector **c** with components $a_1 + b_1$, $a_2 + b_2$, i.e.,

$$\mathbf{a}(a_1, a_2) + \mathbf{b}(b_1, b_2) = \mathbf{c}(a_1 + b_1, a_2 + b_2).$$

For any vectors **a** (a_1, a_2), **b** (b_1, b_2), **c** (c_1, c_2),

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}, \quad \mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}.$$

To prove, it suffices to compare the corresponding components of the vectors on the right- and left-hand sides. We see that they are equal, and vectors with corresponding equal components are equal.

For any three points A, B, C, the vector equation

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

holds.

Proof. Let $A (x_1, y_1)$, $B (x_2, y_2)$, $C (x_3, y_3)$ be the three given points. The components of the vector $\overrightarrow{AB}$ are $(x_2 - x_1, y_2 - y_1)$, whereas those of $\overrightarrow{BC}$, $(x_3 - x_2, y_3 - y_2)$. Therefore, the components of the vector $\overrightarrow{AB} + \overrightarrow{BC}$ are $(x_3 - x_1, y_3 - y_1)$, just those of the vector $\overrightarrow{AC}$. Thus, $\overrightarrow{AB} + \overrightarrow{BC}$ and $\overrightarrow{AC}$ are equal. Q.E.D.

Hence, the following method for the construction of the sum of two arbitrary vectors **a** and **b**. Viz., we have to mark off a vector **b'**

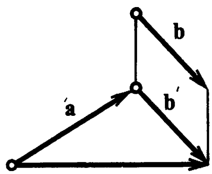

Fig. 20

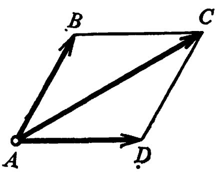

Fig. 21

equal to **b** from the end-point of **a**. Then the vector whose origin coincides with that of **a**, and end-point with that of **b'**, is the sum **a** + **b** (Fig. 20).

Given a parallelogram ABCD, prove the vector equation $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$ *("parallelogram law" of vector addition).*

Solution. We have: $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ (Fig. 21). But the vectors $\overrightarrow{BC}$ and $\overrightarrow{AD}$ are equal. Therefore, $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$.

A vector **c** (c_1, c_2) whose sum with a vector **b** yields a vector **a** is called the *difference* of the vectors **a** (a_1, a_2) and **b** (b_1, b_2), viz., **b** + **c** = **a**. Hence, we find the components of the vector **c** = **a** − **b**, i.e., $c_1 = a_1 - b_1$, $c_2 = a_2 - b_2$.

Given two vectors $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$ *with a common origin, prove that* $\overrightarrow{AC} - \overrightarrow{AB} = \overrightarrow{BC}$.

Solution. We have $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$, which means that $\overrightarrow{AC} - \overrightarrow{AB} = \overrightarrow{BC}$.

5. Multiplication of a Vector by a Number

The *product* of a vector $\overrightarrow{(a_1, a_2)}$ and a number λ is the vector $\overrightarrow{(\lambda a_1, \lambda a_2)}$, viz.,

$$\overrightarrow{(a_1, a_2)}\,\lambda = \lambda\,\overrightarrow{(a_1, a_2)} = \overrightarrow{(\lambda a_1, \lambda a_2)}.$$

It follows from the definition of the multiplication of a vector by a number that, *for any vector* **a** *and two numbers* λ, μ,

$$(\lambda + \mu)\,\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}.$$

For any two vectors **a** *and* **b** *and a number* λ,

$$\lambda\,(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}.$$

The modulus of the vector $\lambda\mathbf{a}$ *equals* $|\lambda|\,|\mathbf{a}|$. *The direction of* $\lambda\mathbf{a}$ *coincides with that of* **a** *if* $\lambda > 0$, *and is opposite if* $\lambda < 0$.

Proof. Construct two vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ equal to **a** and $\lambda\mathbf{a}$, respectively, O being the common origin. Let a_1 and a_2 be the components of **a**. Then the coordinates of the point A are the values a_1 and a_2, whereas those of the point B are λa_1, λa_2.

In the case $0 < \lambda < 1$, B is on the line segment OA, and divides it in the ratio $\lambda : (1 - \lambda)$, since its coordinates admit the representation

$$\lambda a_1 = \frac{0\cdot(1-\lambda)+\lambda a_1}{(1-\lambda)+\lambda}, \quad \lambda a_2 = \frac{0\cdot(1-\lambda)+\lambda a_2}{(1-\lambda)+\lambda}.$$

In the case $\lambda > 1$, A is on the line segment OB, and divides it in the ratio $1 : (\lambda - 1)$. Thus, in both cases, i.e., for $\lambda > 0$, $\overrightarrow{OB}$ has the same direction as $\overrightarrow{OA}$.

In the case $\lambda < 0$, the point O is on the line segment AB, and divides it in the ratio $1 : |\lambda|$. Hence, if $\lambda < 0$, then $\overrightarrow{OB}$ has the direction opposite to that of $\overrightarrow{OA}$, and we obtain

$$|\lambda\mathbf{a}| = \sqrt{(\lambda a_1)^2 + (\lambda a_2)^2} = |\lambda|\sqrt{a_1^2 + a_2^2} = |\lambda|\,|\mathbf{a}|.$$

Q.E.D.

Given two points $A\,(x_1, y_1)$ and $B\,(x_2, y_2)$, *prove that the vectors* $\overrightarrow{AB}$ *and* $\overrightarrow{BA}$ *are opposite.*

Solution. The components of $\overrightarrow{AB}$ are $x_2 - x_1$ and $y_2 - y_1$, whereas those of $\overrightarrow{BA}$ are $x_1 - x_2$ and $y_1 - y_2$. We see that $\overrightarrow{AB} = (-1)\,\overrightarrow{BA}$. Therefore, $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are opposite.

6. Collinear Vectors

Two vectors are said to be *collinear* if they are on the same or parallel straight lines.

The corresponding components of collinear vectors are proportional. Conversely, if the corresponding components are proportional, then the vectors are collinear.

Proof. Let $\mathbf{a}\,(a_1, a_2)$ and $\mathbf{b}\,(b_1, b_2)$ be the given vectors. Assume that they are collinear. Consider the vector $\mathbf{c} = \pm\frac{|\mathbf{a}|}{|\mathbf{b}|}\,\mathbf{b}$, where the plus is taken when **a** and **b** are co-directional, and the minus when they are opposite. The vector **c** equals **a**, since they are co-directional, and equal in modulus. Equalizing their components, we obtain

$$a_1 = \pm \frac{|\mathbf{a}|}{|\mathbf{b}|} b_1, \quad a_2 = \pm \frac{|\mathbf{a}|}{|\mathbf{b}|} b_2.$$

Hence, $\frac{b_1}{a_1} = \pm \frac{|\mathbf{b}|}{|\mathbf{a}|}$, $\frac{b_2}{a_2} = \pm \frac{|\mathbf{b}|}{|\mathbf{a}|}$, and $\frac{b_1}{a_1} = \frac{b_2}{a_2}$, i.e., the components of **a** and **b** are proportional.

Now, let the coordinates of **a** and **b** be proportional. We prove that the vectors are collinear. We have

$$\frac{b_1}{a_1} = \frac{b_2}{a_2}.$$

Denoting the common value of these ratios by λ, we obtain $b_1 = \lambda a_1$, $b_2 = \lambda a_2$. Hence, $\mathbf{b} = \lambda \mathbf{a}$, which means that the vectors are collinear.

Given that the vectors **a** (1, —1) *and* **b** (—2, *m*) *are collinear, find m.*

Solution. The components of collinear vectors are proportional. Therefore, $\frac{-2}{1} = \frac{m}{-1}$, implying that $m = 2$.

7. Resolution of a Vector into Two Non-Collinear Vectors

If two vectors **a** *and* **b** *are other than zero and non-collinear, then any vector* **c** *admits one, and only one, representation*

$$\mathbf{c} = \lambda \mathbf{a} + \mu \mathbf{b}.$$

Proof. If **c** is the zero vector, then $\mathbf{c} = 0 \cdot \mathbf{a} + 0 \cdot \mathbf{b}$. Let **c** be non-zero. Draw straight lines parallel to **a** and **b** through its end-points (Fig. 22). Accordingly, we obtain its representation as the sum of collinear vectors $\mathbf{a}_1$, $\mathbf{b}_1$ and **a**, **b**, respectively. We have

$$\mathbf{a}_1 = \lambda \mathbf{a}, \quad \mathbf{b}_1 = \mu \mathbf{b}.$$

Therefore,

$$\mathbf{c} = \lambda \mathbf{a} + \mu \mathbf{b}.$$

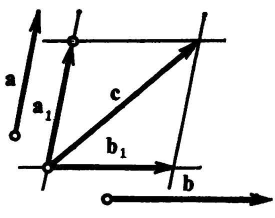

Fig. 22

To prove the representation uniqueness, we suppose that there are two

$$\mathbf{c} = \lambda_1 \mathbf{a} + \mu_1 \mathbf{b}, \quad \mathbf{c} = \lambda_2 \mathbf{a} + \mu_2 \mathbf{b}.$$

Subtracting one of the equalities termwise from the other, we obtain

$$\mathbf{0} = (\lambda_1 - \lambda_2)\mathbf{a} + (\mu_1 - \mu_2)\mathbf{b},$$

which is possible only if $\lambda_1 - \lambda_2 = 0$, $\mu_1 - \mu_2 = 0$, since **a** and **b** are non-collinear, and the uniqueness is thus proved.

A vector is said to be *unit* if its modulus is unity. Unit vectors with the direction of positive coordinate half-axes are called *base vectors*. We will denote them by $\mathbf{e}_1$ (1, 0) on the x-axis and, by $\mathbf{e}_2$ (0, 1) on the y-axis.

Any vector $\mathbf{a}$ (a_1, a_2) *admits a representation in the form*

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2.$$

In fact, $\overrightarrow{(a_1, a_2)} = \overrightarrow{(a_1, 0)} + \overrightarrow{(0, a_2)} = a_1 \overrightarrow{(1, 0)} + a_2 \overrightarrow{(0, 1)} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$.

Given three vectors $\mathbf{a}$ (1, 0), $\mathbf{b}$ (1, 1), $\mathbf{c}$ (—1, 0), *decompose* $\mathbf{c}$ *in terms of* $\mathbf{a}$ *and* $\mathbf{b}$.

Solution. Equalizing the corresponding components of the vectors $\mathbf{c}$ and $\lambda\mathbf{a} + \mu\mathbf{b}$, we obtain two equations $-1 = \lambda\cdot 1 + \mu\cdot 1$ and $0 = \lambda\cdot 0 + \mu\cdot 1$, from which $\mu = 0$, $\lambda = -1$.

8. Scalar Product

The *scalar product* of two vectors $\mathbf{a}$ (a_1, a_2) and $\mathbf{b}$ (b_1, b_2) is the value $a_1b_1 + a_2b_2$. We will employ the same notation for the scalar product as for the product of two numbers. The scalar product $\mathbf{aa}$ is denoted by $\mathbf{a}^2$. It is obvious that $\mathbf{a}^2 = |\mathbf{a}|^2$.

It follows that, *for any vectors* $\mathbf{a}$ (a_1, a_2), $\mathbf{b}$ (b_1, b_2), $\mathbf{c}$ (c_1, c_2),

$$(\mathbf{a} + \mathbf{b})\mathbf{c} = \mathbf{ac} + \mathbf{bc}.$$

In fact, the left-hand side of the equality is $(a_1 + b_1)c_1 + (a_2 + b_2)c_2$, whereas the right-hand side is $a_1c_1 + a_2c_2 + b_1c_1 + b_2c_2$. That they are equal is obvious.

The *angle* between two non-zero vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is the angle BAC. The angle between any two vectors $\mathbf{a}$ and $\mathbf{b}$ is that between any two vectors equal to them, but with the common origin. The angle between two co-directional vectors is assumed to be zero.

The scalar product of two vectors equals the product of their moduli times the cosine of the angle between them.

Proof. Let $\mathbf{a}$ and $\mathbf{b}$ be the two given vectors, and φ the angle between them. We have

$$(\mathbf{a} + \mathbf{b})^2 = (\mathbf{a} + \mathbf{b})(\mathbf{a} + \mathbf{b}) = (\mathbf{a} + \mathbf{b})\mathbf{a} + (\mathbf{a} + \mathbf{b})\mathbf{b}$$
$$= \mathbf{aa} + \mathbf{ba} + \mathbf{ab} + \mathbf{bb} = \mathbf{a}^2 + 2\mathbf{ab} + \mathbf{b}^2,$$

or

$$|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{ab}.$$

Hence, the scalar product $\mathbf{ab}$ is expressed in terms of the lengths of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} + \mathbf{b}$, and, therefore, does not depend on the choice of a system of coordinates, i.e., is unaltered if the coordinate system is selected in a special way. Take a coordinate system xy as in Fig. 23. Then, the components of **a** are $|\mathbf{a}|$ and 0, whereas those of **b** are $|\mathbf{b}|\cos\varphi$ and $|\mathbf{b}|\sin\varphi$, and the scalar product is

$$\mathbf{ab} = |\mathbf{a}|\,|\mathbf{b}|\cos\varphi + 0\cdot|\mathbf{b}|\sin\varphi = |\mathbf{a}|\,|\mathbf{b}|\cos\varphi.$$

It follows that *if two vectors are perpendicular, then their scalar product is zero. Conversely, if the scalar product of two non-zero vectors is zero, then the vectors are perpendicular.*

Apply the scalar-product technique to the proof of the Stewart theorem from elementary geometry, viz., *let D be a point on the side*

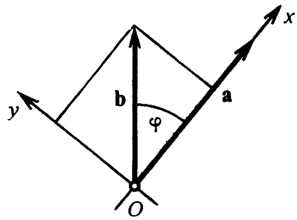

Fig. 23

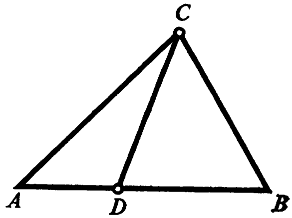

Fig. 24

AB of a triangle ABC (Fig. 24). *Then* $AC^2\cdot BD + BC^2\cdot AD - CD^2\cdot AB = AB\cdot AD\cdot BD$.

Proof. We have the vector equations

$$\overrightarrow{CA} = \overrightarrow{CD} + \overrightarrow{DA}, \quad \overrightarrow{CB} = \overrightarrow{CD} + \overrightarrow{DB}.$$

Squaring them scalarly, we obtain

$$AC^2 = CD^2 + DA^2 + 2\overrightarrow{CD}\cdot\overrightarrow{DA},$$

$$BC^2 = CD^2 + DB^2 + 2\overrightarrow{CD}\cdot\overrightarrow{DB}.$$

Multiplying the first by BD, the second one by AD, and adding termwise, we have

$$AC^2\cdot BD + BC^2\cdot AD$$

$$= (CD^2\cdot BD + CD^2\cdot AD) + (AD^2\cdot BD + BD^2\cdot AD)$$

$$+ 2\,(\overrightarrow{CD}\cdot\overrightarrow{DA}\cdot BD + \overrightarrow{CD}\cdot\overrightarrow{DB}\cdot AD).$$

Since $AD + DB = AB$, the first bracket on the right-hand side equals $CD^2\cdot AB$, and the second one $AB\cdot AD\cdot BD$, whereas the third $2\overrightarrow{CD}\,(\overrightarrow{DA}\cdot BD + \overrightarrow{DB}\cdot AD) = 0$, because the vectors $\overrightarrow{DA}\cdot BD$ and $\overrightarrow{DB}\cdot AD$ are equal in modulus, but are of opposite directions.

Thus, we obtain the equality

$$AC^2\cdot BD + BC^2\cdot AD - CD^2\cdot AB = AB\cdot AD\cdot BD.$$

Q.E.D.

EXERCISES TO CHAPTER II

1. A translation is given by the formulas $x' = x + 1, y' = y - 1$. What points are (0, 0) (1, 0) and (0, 2) sent into?

2. Find a and b in the translation formulas $x' = x + a$, $y' = y + b$, given that (1, 2) is carried into (3, 4).

3. Is there a translation under which the point (1, 2) is sent into (3, 4), whereas (0, 1) into (—1, 0)?

4. Given that AB and CD are two parallel straight lines, and the points B and D are on the same side of a secant AC, prove that the rays AB and CD are co-directional.

5. Prove that the rays AB and CD in Ex. 4 are opposite if the points B and D are on opposite sides of the secant AC.

6. Given three points A, B and C on a straight line, B lying in between A and C, point out the co-directional and opposite vectors among $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

7. Prove that $|\overrightarrow{AC}| \leqslant |\overrightarrow{AB}| + |\overrightarrow{BC}|$ holds for three vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$.

8. Prove that the inequality $|\mathbf{a} + \mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|$ holds for any two vectors $\mathbf{a}$ and $\mathbf{b}$.

9. Given four points A (0, 1), B (1, 0), C (1, 2), D (2, 1), prove that the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal.

10. Given that the modulus of a vector $\mathbf{a}$ (5, m) is 13, find m.

11. Find the modulus of a vector $\mathbf{a} + \mathbf{b}$ if $\mathbf{a} = (1, -4)$, $\mathbf{b} = (-4, 8)$.

12. Show that the sum of n vectors with a common origin at the centre of a regular n-gon, and end-points at its vertices, is zero.

13. Given three vectors with one origin O, and the end-points at the vertices of a triangle ABC, show that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \mathbf{0}$ if and only if O is the point where the three medians meet.

14. Given a vector $\mathbf{r}_{mn}$ with the origin at a point (x_0, y_0) and the end-point at $(m\delta, n\delta)$, where m and n are two integers not exceeding in modulus M and N, respectively, find the sum of all $\mathbf{r}_{mn}$, expressing it in terms of the vector $\mathbf{r}$ with the origin at the point (0, 0) and the end-point at (x_0, y_0).

15. Given a finite figure F in the xy-plane with its centre of symmetry at the origin of coordinates, show that the sum of vectors with a common origin and end-points in the integral points of F, i.e., whose coordinates are integers, is zero if and only if the origin of coordinates is the common origin for all the vectors.

16. Prove that the vectors $\mathbf{a}$ (1, 2) and $\mathbf{b}$ (0.5, 1) are co-directional, whereas the vectors $\mathbf{c}$ (—1, 2) and $\mathbf{d}$ (0.5, —1) are opposite.

17. Given the vector **a** (3, 4), find a vector **b** co-directional with **a**, but of double length.

18. Solve Ex. 17 for a vector **b** opposite to a.

19. Find the modulus of the vector $-2\mathbf{a} +$ for **a** (3, 2) and **b** (0, −1).

20. Given that the modulus of a vector $\lambda\mathbf{a}$ is 5, find λ if **a** has the components (−6, 8).

21. Given the vectors **a** (2, −4), **b** (1, 2), **c** (1, −2), **d** (−2, −4), point out pairs of collinear ones among them.

22. Which vectors are co-directional in Ex. 21, and which are opposite? Which of them are equal in modulus?

23. For what value of n are the vectors **a** $(n, 1)$, **b** $(4, n)$ collinear and co-directional?

24. Find unit vectors among **a** $\left(-\frac{3}{5}, \frac{4}{5}\right)$, **b** $\left(\frac{2}{3}, \frac{2}{3}\right)$, **c** $(0, -1)$, **d** $\left(\frac{3}{5}, -\frac{4}{5}\right)$ and point out collinear ones among them.

25. Find a unit vector collinear and co-directional with the vector **a** (6, 8).

26. Given two midpoints M and N of two line segments AB and CD, respectively, prove the vector equation $\overrightarrow{MN} = \frac{1}{2}(\overrightarrow{AC} + \overrightarrow{BD})$.

27. Given the base vectors $\mathbf{e}_1$ (1, 0) and $\mathbf{e}_2$ (0, 1), what are the components of the vector $2\mathbf{e}_1 - 3\mathbf{e}_2$?

28. What are the values of λ and μ in the representation $\mathbf{a} = \lambda\mathbf{e}_1 + \mu\mathbf{e}_2$ of the vector **a** (−5, 4)?

29. Prove the inequality $(\mathbf{a}\cdot\mathbf{b})^2 \leqslant \mathbf{a}^2\mathbf{b}^2$ for two vectors **a** and **b**.

30. Find the angle between the vectors **a** (1, 2), **b** $\left(1, -\frac{1}{2}\right)$.

31. Given two vectors **a** and **b**, find the modulus of the vector $\mathbf{a} + \mathbf{b}$ if their moduli are unity, and the angle between them is 60°.

32. Find the angle between **a** and $(\mathbf{a} + \mathbf{b})$ from the previous exercise.

33. Given the vertices A (1, 1), B (4, 1) and C (4, 5) of a triangle, find the cosines of its angles.

34. Find the angles of a triangle with the vertices $A(0, \sqrt{3})$, $B(2, \sqrt{3})$, $C\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$.

35. Prove that two vectors **a** (m, n) and **b** $(-n, m)$ are either perpendicular or zero.

36. Given two vectors **a** (3, 4) and **b** $(m, 2)$, for what value of m are they perpendicular?

37. Given the vectors **a** (1, 0) and **b** (1, 1), find λ such that the vector $\mathbf{a} + \lambda\mathbf{b}$ is perpendicular to **a**.

38. For what value of λ is $\mathbf{a} + \lambda\mathbf{b}$ in Ex. 37 perpendicular to **b**?

39. Prove that if **a** and **b** are two unit non-collinear vectors, then the vectors **a** + **b** and **a** — **b** are other than zero, and perpendicular.

40. Given that two unit vectors **a** and **b** form an angle of 60°, prove that the vector 2**b** — **a** is perpendicular to **a**.

41. Given that two vectors **a** + **b** and **a** — **b** are perpendicular, prove that $|\mathbf{a}| = |\mathbf{b}|$.

42. Given four points A (1, 1), B (2, 3), C (0, 4) and D (—1, 2), prove that the quadrilateral $ABCD$ is a rectangle.

43. Given four points A (0, 0), B (—1, 1), C (0, 2) and D (1, 1), prove that the quadrilateral $ABCD$ is a square.

44. Prove that if **a** and **b** are arbitrary non-zero and non-collinear vectors, then

$$\lambda^2\mathbf{a}^2 + 2\lambda\mu\,(\mathbf{ab}) + \mu^2\mathbf{b}^2 \geqslant 0,$$

equality holding only if $\lambda = \mu = 0$.

Chapter III

STRAIGHT LINE IN THE PLANE

1. Equation of a Straight Line. General Form

Let us prove that *any straight line in the xy-plane is described by an equation of the form*

$$ax + by + c = 0 \qquad (*)$$

where a, b, c are constants. Conversely, *any equation of the form* (*) *is the equation of a straight line.*

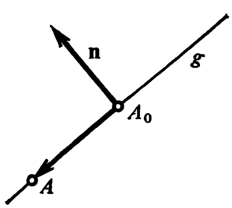

Fig. 25

Proof. Let g be an arbitrary straight line, A_0 (x_0, y_0) a point on it and **n** (a_1, a_2) a vector perpendicular to g (Fig. 25). Let then A (x, y) be an arbitrary point on the line. Vectors $\overrightarrow{A_0A}$ and **n** will then be perpendicular, and hence their scalar product will be zero. Thus, each point on g will obey

$$(x - x_0)a_1 + (y - y_0)a_2 = 0. \qquad (**)$$

Conversely, if A (x, y) satisfies this equation, then this means that $\overrightarrow{A_0A} \cdot \mathbf{n} = 0$, and hence A lies on g.

By definition, the equation (**) is the equation of g. It can be rewritten as

$$a_1x + a_2y + (-a_1x_0 - a_2y_0) = 0.$$

We see that it has the form (*). This proves the first statement.

Suppose now we have the equation

$$ax + by + c = 0.$$

We will now see that it is the equation of a certain straight line. Let x_0, y_0 be some solution of that equation, i.e.

$$ax_0 + by_0 + c = 0.$$

Using this relationship we can transform our equation as follows:

$$ax + by - ax_0 - by_0 = 0,$$

or

$$a(x - x_0) + b(y - y_0) = 0.$$

But in this form, as we have seen, it is the equation of the straight line passing through (x_0, y_0) at right angles to $\mathbf{n}$ (a, b). We have thus proved the second statement as well.

Note that *in the equation of the straight line*

$$ax + by + c = 0$$

the coefficients a and b are the coordinates of the vector perpendicular to the straight line.

By way of exercise, we now form the equation of a straight line passing through two given points (x_1, y_1) and (x_2, y_2).

The vector $\mathbf{e}$ $(x_2 - x_1, y_2 - y_1)$ lies on the desired straight line. The vector $\mathbf{e}'$ $(y_1 - y_2, x_2 - x_1)$ is perpendicular to $\mathbf{e}$, since the scalar product $\mathbf{ee}' = 0$. This means that $\mathbf{e}'$ is perpendicular to the straight line. And then, as we already know, the equation of the straight line can be written as

$$(x - x_1)(y_1 - y_2) + (y - y_1)(x_2 - x_1) = 0.$$

This equation is more easily remembered when written as:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}.$$

Examine the locus of points such that the difference of squared distances from two given points is constant.

Solution. Let (x_0, y_0) and (x_0', y_0') be the given points and (x, y) an arbitrary point of the locus. We have

$$(x - x_0)^2 + (y - y_0)^2 - (x - x_0')^2 - (y - y_0')^2 = c = \text{const},$$

or

$$2x(x_0' - x_0) + 2y(y_0' - y_0) + x_0^2 + y_0^2 - x_0'^2 - y_0'^2 - c = 0.$$

We see that our equation is linear, i.e., of the first degree in x and y, and with constant coefficients. Consequently, our locus is a straight line.

2. Position of a Straight Line Relative to a Coordinate System

We will now analyze the position of a straight line relative to a coordinate system if the line is described by some special form of the equation $ax + by + c = 0$.

(1) $a = 0$. The equation becomes

$$y = -\frac{c}{b}.$$

All the points on the line have thus the same ordinate $(-c/b)$, and hence the *straight line is parallel to the x-axis* (Fig. 26*a*). Specifically, *if also* $c = 0$, *the straight line coincides with the x-axis.*

(2) $b = 0$. The case is similar to the first one. *The straight line is parallel to the y-axis* (Fig. 26*b*) *and coincides with it if* $c = 0$.

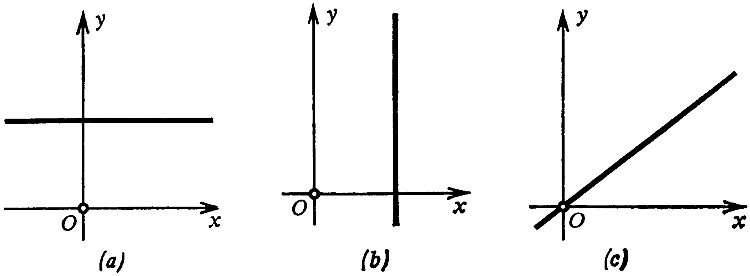

Fig. 26

(3) $c = 0$. *The straight line passes through the origin of coordinates,* since (0, 0) satisfies the equation of a straight line (Fig. 26*c*).

If in the equation $ax + by + c = 0$ the coefficient at y is nonzero, then this equation can be solved relative to y. We get

$$y = -\frac{a}{b}x - \frac{c}{b}$$

Or, denoting $-a/b = k$, $-c/b = q$, we obtain

$$y = kx + q.$$

We now clarify the geometric meaning of the coefficient k in that equation. We take on the straight line two points $A(x_1, y_1)$, $B(x_2, y_2)$, $x_1 < x_2$. Their coordinates obey the equation of a straight line:

$$y_1 = kx_1 + q, \quad y_2 = kx_2 + q.$$

Subtracting these equalities termwise, we will obtain $y_2 - y_1 = k(x_2 - x_1)$. Hence

$$k = \frac{y_2 - y_1}{x_2 - x_1}.$$

In the case shown in Fig. 27*a*, $\frac{y_2-y_1}{x_2-x_1}=\tan\alpha$. In the case in Fig. 27*b*, $\frac{y_2-y_1}{x_2-x_1}=-\tan\alpha$. Accordingly, the coefficient k in the equation of a straight line is equal, up to a sign, to the tangent of the acute angle formed by the straight line with the x-axis.

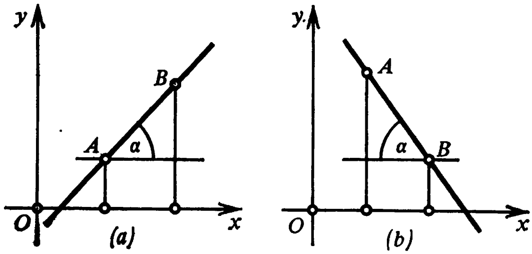

Fig. 27

It is generally said that the *coefficient k in the equation of a straight line is equal to the tangent of the angle that the line makes with the x-axis*. This angle is taken to be negative in Fig. 27*b*. In the equation k is called the *slope of the line*.

3. Parallelism and Perpendicularity Condition for Straight Lines

Consider the equations of two straight lines:

$$a_1x + b_1y + c_1 = 0,$$
$$a_2x + b_2y + c_2 = 0.$$

We now examine the conditions to be met by the coefficients of the equations of straight lines for the lines to be parallel (perpendicular).

As we know now, the coefficients of x and y in the equation of a straight line are the coordinates of the vector perpendicular to the line. Therefore, for the lines to be parallel it is necessary and sufficient that the vectors perpendicular to them be collinear. Hence the parallelism condition for straight lines,

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}.$$

For the straight lines to be perpendicular it is necessary and sufficient for the vectors perpendicular to them to be perpendicular to each other, and hence their scalar product is zero. Hence, the perpendicularity condition for straight lines,

$$a_1a_2 + b_1b_2 = 0.$$

A straight line is specified by its equation $ax + by + c = 0$ *and a point* (x_0, y_0). *Form the equation of the straight line passing through* (x_0, y_0) *parallel (perpendicular) to the given straight line.*

Solution. To begin with, we form the equation of the parallel straight line. Since the desired straight line is parallel to the given one, the vector $\overrightarrow{(a, b)}$ perpendicular to the given straight line will be perpendicular to the desired straight line. Knowing a point and a vector perpendicular to the straight line, we obtain its equation

$$a(x - x_0) + b(y - y_0) = 0.$$

We now find the equation of the straight line perpendicular to the given one. The vector $\overrightarrow{(a, b)}$ is perpendicular to the given straight line. The vector $\overrightarrow{(-b, a)}$ is perpendicular to the vector $\overrightarrow{(a, b)}$, since their scalar product is zero. Therefore, the vector $\overrightarrow{(-b, a)}$ is perpendicular to the desired straight line. And now we can readily write the equation

$$-b(x - x_0) + a(y - y_0) = 0.$$

Consider two intersecting straight lines given by their equations

$$ax + by + c = 0 \quad \text{and} \quad a_1x + b_1y + c_1 = 0.$$

Find the angle between them.

Solution. The angle between straight lines, by definition, is the least angle formed by the intersection of two lines. This angle equals that between the vectors perpendicular to the straight lines, or supplements it to 180°. Therefore, the cosine of the angle between the straight lines equals, up to a sign, the cosine of the angle between $\overrightarrow{(a, b)}$ and $\overrightarrow{(a_1, b_1)}$. Using the scalar product of the vectors, we obtain the equation for the angle φ between the straight lines

$$|aa_1 + bb_1| = \sqrt{a^2 + b^2}\,\sqrt{a_1^2 + b_1^2}\cos\varphi.$$

Hence we find φ $(0 < \varphi \leqslant \pi/2)$.

4. Equation of a Pencil of Straight Lines

Consider two intersecting straight lines given by the equations

$$ax + by + c = 0,$$
$$a_1x + b_1y + c_1 = 0.$$

We will write the equation

$$\lambda(ax + by + c) + \mu(a_1x + b_1y + c_1) = 0 \qquad (*)$$

where λ and μ are constants. This equation is linear, and so is the equation of some straight line. The coordinates of the intersection point of the given straight lines obey this equation, since for them $ax + by + c = 0, a_1x + b_1y + c_1 = 0$. The equation (*) is called the *equation of a pencil of straight lines.*

The equation of a pencil comes in handy when constructing the equation of a straight line passing through the point of intersection of the two given straight lines and meeting some additional condition. By way of illustration we solve the following problem.

Consider the equations of two intersecting straight lines $ax + by + c = 0$, $a_1x + b_1y + c_1 = 0$. *Find the equation of the straight line passing through a given point* (x_0, y_0) *and the intersection point of the given straight lines.*

Solution. *The straight line specified by the equation* $\lambda (ax + by + c) + \mu (a_1x + b_1y + c_1) = 0$ passes through the point of intersection of the given straight lines. We require that it should pass also through (x_0, y_0). To this end, it is necessary that $\lambda (ax_0 + by_0 + c) + \mu (a_1x_0 + b_1y_0 + c_1) = 0$. For any λ and μ that are not zero simultaneously and meet this equation, a straight line belonging to the pencil will pass through (x_0, y_0). Specifically, we can take $\lambda = a_1x_0 + b_1y_0 + c_1$, $\mu = -(ax_0 + by_0 + c)$. Then the equation of the desired straight line will be

$$(ax + by + c)(a_1x_0 + b_1y_0 + c) - (a_1x + b_1y + c_1)(ax_0 + by_0 + c) = 0.$$

5. Normal Form of the Equation of a Straight Line

The equation of a straight line $ax + by + c = 0$ is said to be in normal form, if $a^2 + b^2 = 1$. It is obvious that for general form of the equation of a straight line to be reduced to normal form it is sufficient to divide it by $\pm\sqrt{a^2 + b^2}$.

The equation of a straight line in normal form has a simple geometrical meaning. Namely, if into its left-hand side we substitute the coordinates of any point in the plane, then we will obtain a number that, except for sign, will equal the distance from the point to the straight line. And for points in one half-plane defined by the straight line this number is positive, and for the other negative. We will now prove this.

Let

$$ax + by + c = 0$$

be the equation of a straight line in normal form, and $A_0 (x_0, y_0)$ be some point on the straight line. Then $ax_0 + by_0 + c = 0$, and the equation of the straight line can be represented as

$$a\ (x - x_0) + b\ (y - y_0) = 0.$$

If now $A_1\ (x_1, y_1)$ is an arbitrary point in the plane, then substituting its coordinates into the left-hand side of the equation of the straight line gives

$$a\ (x_1 - x_0) + b\ (y_1 - y_0) = \mathbf{n}\cdot\overrightarrow{A_0A_1},$$

where $\mathbf{n}\ (a, b)$ is the unit vector $(a^2 + b^2 = 1)$. We have $|\mathbf{n}\cdot\overrightarrow{A_0A_1}| = A_0A_1\cdot\cos\theta = A_1B$ (Fig. 28). In fact, substitution of the coordinates of A_1 into the left-hand side of the equation yields, up to a sign, the distance A_1B of point A_1 from the straight line.

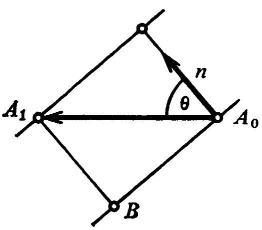

Fig. 28

Clearly, the sign of the expression $\mathbf{n}\cdot\overrightarrow{A_0A_1}$ depends on the directions of $\mathbf{n}$ and $\overrightarrow{A_0A_1}$, whether they are directed into the same half-plane or not. Therefore, for points in one half-plane the expression is positive, and for the other negative.

Examine the locus of points equidistant from two intersecting straight lines.

Solution. Consider $ax + by + c = 0$ and $a_1x + b_1y + c_1 = 0$—two equations of straight lines in normal form. If we substitute the coordinates of an arbitrary point into the equations, we obtain, up to a sign, the distances from that point to the straight lines. It follows that the points of the desired locus obey the equation

$$|\ ax + by + c\ | = |\ a_1x + b_1y + c_1\ |.$$

This equation is equivalent to the two equations

$$ax + by + c = a_1x + b_1y + c_1,$$
$$ax + by + c = -\ (a_1x + b_1y + c_1).$$

Consequently, the desired locus consists of two straight lines. Clearly, these are the lines including the bisectors of the angles obtained when the two straight lines intersect.

6. Transformation of Coordinates

Consider xy- and $x'y'$-coordinate systems in a plane (Fig. 29). We now establish the relation between the coordinates of an arbitrary point relative to these coordinate systems.

Let

$$a_1x + b_1y + c_1 = 0,$$
$$a_2x + b_2y + c_2 = 0$$

be the equations of the y'- and x'-axes in normal form in the xy-coordinate system.

The equation of a straight line in normal form is defined uniquely up to a sign of all the coefficients of the equation. Therefore, we can consider without loss of generality that for a certain point A_0 (x_0, y_0) in the first quadrant of the $x'y'$-coordinate system we have

$$a_1x_0 + b_1y_0 + c_1 > 0,$$
$$a_2x_0 + b_2y_0 + c_2 > 0$$

(in the opposite case we can change the sign of the coefficients).

Statement. *The coordinates of an arbitrary point x', y' relative to the $x'y'$-coordinate system are expressed through the coordinates x, y of the same point in the xy-coordinate system by*

$$x' = a_1x + b_1y + c_1, \qquad (*)$$
$$y' = a_2x + b_2y + c_2.$$

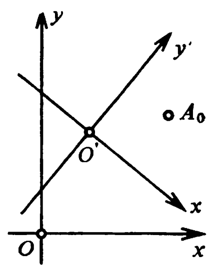

Fig. 29

We prove, for example, the first of these. The right- and left-hand sides of the equation are equal in absolute value, since this value is the distance from the point to the y'-axis. In each of the half-planes defined by the y'-axis, the right- and left-hand sides of the formula retain the sign and change it in passing from one half-plane to the other. And since for A_0 the signs coincide, they coincide for any point in the plane.

The second formula is proved in an analogous fashion.

Since

$$a_1x + b_1y + c_1 = 0,$$
$$a_2x + b_2y + c_2 = 0$$

are the equations of two intersecting straight lines in normal form, then the coefficients a_1, b_1, a_2, b_2 in $(*)$ are related by

$$a_1^2 + b_1^2 = 1, \qquad (**)$$
$$a_2^2 + b_2^2 = 1,$$
$$a_1a_2 + b_1b_2 = 0.$$

If we take into consideration the first two of $(**)$, we can represent a_1, b_1, a_2, b_2 as

$$a_1 = \cos\alpha, \quad b_1 = \sin\alpha,$$
$$a_2 = \cos\alpha_1, \quad b_2 = \sin\alpha_1.$$

From the third of $(**)$ we then obtain

$$\cos\alpha\cos\alpha_1 + \sin\alpha\sin\alpha_1 = \cos(\alpha - \alpha_1) = 0,$$

whence $\alpha_1 = \alpha \pm \pi/2$. Thus, we can write the formulas of coordinate transformation (*) in one of the following two forms

$$x' = x \cos \alpha + y \sin \alpha + c_1,$$
$$y' = -x \sin \alpha + y \cos \alpha + c_2$$

or

$$x' = x \cos \alpha + y \sin \alpha + c_1,$$
$$y' = x \sin \alpha - y \cos \alpha + c_2.$$

The first of these covers all the cases when the $x'y'$-coordinate system can be obtained by a continuous motion from the xy-coordinate system. The second system of formulas embraces the cases when the $x'y'$-coordinate system is obtained from the xy-coordinate system by a motion and a mirror reflection.

Quantities α, c_1 and c_2 in the transformation formulas have a simple geometrical meaning: α is, up to the multiple of 2π, the angle formed by the x'-axis with the x-axis, and c_1 and c_2 are the coordinates of the origin of the xy-coordinate system in the $x'y'$-coordinate system.

In the xy-plane a new x'y'-coordinate system is introduced. The coordinate axes of the new system in the xy-coordinate system are given by the equations

$$3x + 4y + 10 = 0,$$
$$-4x + 3y - 15 = 0.$$

Find the formulas for sending x, y to x', y', given that the old origin lies in the first quadrant in the new system.

Solution. We transform the equations of the new axes to normal form to get

$$\frac{3}{5}x + \frac{4}{5}y + 2 = 0, \quad -\frac{4}{5}x + \frac{3}{5}y - 3 = 0.$$

The transformation formulas are known to have the form

$$x' = \pm\left(\frac{3}{5}x + \frac{4}{5}y + 2\right), \quad y' = \pm\left(-\frac{4}{5}x + \frac{3}{5}y - 3\right).$$

The choice of sign of the right-hand side of the formulas is determined by the fact that the origin of the old coordinate system lies in the first quadrant in the new system. Hence, substitution of $x = 0$ and $y = 0$ into the right-hand sides of formulas must yield positive values. To this end, in the first formula we should take the plus sign, and in the second the minus sign.

7. Motions in the Plane

The formulas of transformation of coordinates

$$x' = x \cos \alpha + y \sin \alpha + c_1, \quad (*)$$
$$y' = \pm (-x \sin \alpha + y \cos \alpha + c_2)$$

have another important interpretation. Take in the plane any two points A_1 and A_2. The distance between them can be expressed in the xy-coordinate system and in the $x'y'$-coordinate system. We have

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = (x_2' - x_1')^2 + (y_2' - y_1')^2. \quad (**)$$

Now consider a transformation of the xy-plane such that an arbitary point (x, y) in it changes into a point (x', y') according to the formulas (*). It follows from (**) that this transformation of the plane is motion.

It is easily seen that any motion in the xy-plane is given by formulas of the form (*). Namely, if a given motion transforms the xy-coordinate system into the $x'y'$-coordinate system, then the transformation formulas will be the formulas describing this motion.

The motion given by (*) can be obtained from the elementary motion given by

$$x' = x \cos \alpha + y \sin \alpha, \quad (***)$$
$$y' = \pm (-x \sin \alpha + y \cos \alpha),$$

and the transition $x' = x + c_1$, $y' = y + c_2$.

The motion given by (***) with the plus sign is, by the second formula, a rotation about the origin of coordinates. The motion given by (***) with the minus sign is, by the second formula, a mirror reflection, i.e. there is a symmetry about some straight line passing through the origin of coordinates.

8. Inversion

Let O be an arbitrary point in a plane and R a positive number. A transformation under which any point X, other than O, shifts to a point X' on the OX ray, such that $OX \cdot OX' = R^2$, is called the *inversion*. The point O is called the *centre of inversion*, and R the *radius of inversion*. Clearly, the inversion shifts X' to X.

Inversion can be visualized as follows. We draw a circle with centre O and radius R (Fig. 30). If a point X lies beyond the circle, then to obtain X' it is necessary to draw a tangent to the circle from X and from the point of tangency to drop a perpendicular onto the straight line OX. The foot of this perpendicular will be X'. In fact, it is common knowledge that in a right-angled triangle OAX we have $OA^2 = OX' \cdot OX$.

If a point X lies within the circle, we must draw through it a chord perpendicular to OX, and at the end of the chord construct the tangent to the circle. The intersection of the tangent with OX will yield X'.

If a point X lies on the circle, then X' coincides with X.

Under inversion, a circle that does not pass through the centre of inversion becomes a circle; a circle that passes through the centre of inversion becomes a straight line; a straight line that does not pass through the centre of inversion becomes a circle that passes through the centre of inversion; a straight line that passes through the centre inverts into itself.

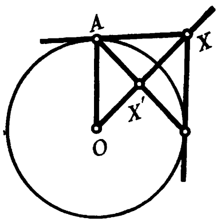

Fig. 30

Proof. We take the centre of inversion O to be the origin of the xy-coordinate system. We will express the coordinates x, y of point X through the coordinates x', y' of the point X', into which X changes under inversion. Since vectors $\vec{OX}$ and $\vec{OX'}$ are collinear, $x = \lambda x'$, and $y = \lambda y'$. Since $OX' \cdot OX = R^2$, $(x'^2 + y'^2)((\lambda x')^2 + (\lambda y')^2) = R^4$. Hence $\lambda = R^2/(x'^2 + y'^2)$. Thus,

$$x = \frac{R^2 x'}{x'^2 + y'^2}, \qquad y = \frac{R^2 y'}{x'^2 + y'^2}. \qquad (*)$$

Now we take an arbitrary circle. It is given by an equation of the form

$$x^2 + y^2 + ax + by + c = 0.$$

Substituting the expressions for x and y given by (*), we will get the equation of the curve into which the circle changes under inversion

$$\frac{1}{x'^2 + y'^2}(R^4 + aR^2x' + bR^2y' + c(x'^2 + y'^2)) = 0.$$

The curve given by the equation

$$R^4 + aR^2x + bR^2y + c(x^2 + y^2) = 0$$

for $c \neq 0$, as we know, is a circle. Thus, an inversion changes a circle that does not pass through the centre of inversion into a circle.

If a circle passes through the centre of inversion ($c = 0$), it transforms into the straight line

$$R^4 + aR^2x + bR^2y = 0.$$

The straight line $ax + by + c = 0$ that does not pass through the centre of inversion ($c \neq 0$) changes into the circle

$$aR^2x + bR^2y + c(x^2 + y^2) = 0$$

that passes through the centre of inversion (the origin of coordinates).

The straight line $ax + by = 0$ that passes through the centre of inversion, is sent into the straight line $aR^2x + bR^2y = 0$, i.e. into itself. This completes the proof.

EXERCISES TO CHAPTER III

1. Form the equation of the locus of points equidistant from two points (0, 1) and (1, 2).

2. Find the points where the straight line $x + 2y + 3 = 0$ cuts the axes of coordinates.

3. Find the point of intersection of the straight lines $x + 2y + 3 = 0$ and $4x + 5y + 6 = 0$.

4. Form the equation of the straight line that passes through points A (—1, 1) and B (1, 0).

5. Find the coefficients a and b in the equation $ax + by = 1$, given that it passes through points (1, 2) and (2, 1).

6. Find the coefficient c in the equation $x + y + c = 0$, given that it passes through the point (1, 2).

7. Find the value of c at which the straight line $x + y + c = 0$ touches the circle $x^2 + y^2 = 1$.

8. Prove that the three straight lines $x + 2y = 3$, $2x - y = 1$, and $3x + y = 4$ meet at one point.

9. Prove that the straight lines $x + 2y = 3$ and $2x + 4y = 3$ do not intersect.

10. From the equation of a straight line, knowing that it is parallel to the x-axis and passes through the point (2, 3).

11. Form the equation of a straight line, knowing that it passes through the origin of coordinates and the point (2, 3).

12. Form the equation of the straight line that passes through the point (x_0, y_0) and is equidistant from points (x_1, y_1) and (x_2, y_2).

13. Show that the three points (x_1, y_1), (x_2, y_2), (x_3, y_3) lie on a straight line if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

14. Show that the equation

$$a^2x^2 + 2abxy + b^2y^2 - c^2 = 0$$

defines a pair of straight lines. Find the equation for each of the lines.

15. Show that any straight line can be defined in a parametric way by equations of the form

$$x = at + b, \quad y = ct + d \quad (-\infty < t < \infty).$$

Conversely, any such system of equations can be viewed as equations of a certain straight line in parametric form. This line is given by an equation in implicit form

$$(x - b) c - (y - d) a = 0.$$

16. A curve γ is given by the equation

$$\omega (x, y) = 0,$$

where ω is a polynomial of degree n in x and y. Show that if the curve γ has with some straight line more than n intersection points, then it includes this straight line completely.

17. The locus of points of equal powers with respect to two circles (see Exercise 40 to Chapter 2) is called the *radical axis* of two circles. Show that the radical axis is a straight line. If the circles intersect, then it passes through the intersection points.

18. Under what condition does the straight line

$$ax + by + c = 0$$

intersect the positive x-axis (negative x-axis)?

19. Under what condition is the straight line

$$ax + by + c = 0$$

not in the first quadrant?

20. Show that the straight lines given by

$$ax + by + c = 0 \quad \text{and} \quad ax - by + c = 0 \quad (b \neq 0)$$

are symmetric about the x-axis.

21. Show that the straight lines given by

$$ax + by + c = 0 \quad \text{and} \quad ax + by - c = 0,$$

are symmetric about the origin of coordinates.

22. Consider the pencil of straight lines

$$ax + by + c + \lambda (a_1x + b_1y + c_1) = 0.$$

Find the value of λ at which a line in the pencil is parallel to the x-axis (y-axis) and at which it passes through the origin of coordinates.

23. Under what condition does the straight line

$$ax + by + c = 0$$

and the coordinate axes bound an isosceles triangle?

24. Show that the area of the triangle bounded by the straight line

$$ax + by + c = 0 \quad (a, b, c \neq 0)$$

and the coordinate axes is

$$S = \frac{1}{2} \frac{c^2}{|ab|}.$$

25. Find the tangents to the circle

$$x^2 + y^2 + 2ax + 2by = 0$$

parallel to the coordinate axes.

26. Show that the straight lines $ax + by + c = 0$, $bx - ay + c' = 0$ meet at right angles.

27. Find the angle formed by the x-axis and the straight line

$$y = x \cot \alpha \quad \left(-\frac{\pi}{2} < \alpha < 0\right).$$

28. Form the equations of the sides of an equilateral triangle with side 1, with one of the sides and the altitude dropped on it as the coordinate axes.

29. Find the interior angles of the triangle bounded by the straight lines

$$x + 2y = 0, \quad 2x + y = 0, \quad \text{and} \quad x + y = 1.$$

30. Under what condition is the x-axis the bisector of the angles formed by the straight lines

$$ax + by = 0, \quad a_1x + b_1y = 0?$$

31. For the angle θ formed by the straight line

$$x = at + b, \quad y = ct + d$$

with the x-axis derive the formula

$$\tan \theta = \frac{c}{a}.$$

32. Find the angle between the straight lines given by the equations in parametric form

$$x = a_1t + b_1, \qquad x = c_1t + d_1,$$

and

$$y = a_2t + b_2 \qquad y = c_2t + d_2.$$

33. Show that the quadrilateral bounded by the straight lines

$$\pm ax \pm by + c = 0 \quad (a, b, c \neq 0),$$

is a rhombus. The coordinate axes are its diagonals.

34. Show that two straight lines that cut off on the coordinate axes sections of equal length are either parallel or perpendicular.

35. Find the parallelism (perpendicularity) condition for the straight lines given by the equations in parametric form

$$x = \alpha_1t + a_1, \qquad x = \alpha_2t + a_2,$$

and

$$y = \beta_1t + b_1 \qquad y = \beta_2t + b_2.$$

36. Find the parallelism (perpendicularity) condition for the straight lines one of which is given by the equation

$$ax + by + c = 0,$$

and the other by the equations in parametric form

$$x = \alpha t + \beta, \quad y = \gamma t + \delta.$$

37. In the family of straight lines given by the equations

$$a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$$

(λ is the parameter of the family), find the line parallel (perpendicular) to the straight line $ax + by + c = 0$.

38. Given the equations of the sides of a triangle and the coordinates of a point, think of the way of finding out whether or not this point lies within the triangle.

39. Show that the distance between the parallel straight lines

$$ax + by + c_1 = 0, \quad ax + by + c_2 = 0$$

is

$$\frac{|c_1 - c_2|}{\sqrt{a^2+b^2}}.$$

40. Form the equations of the straight lines parallel to the line

$$ax + by + c = 0,$$

that are separated from it by δ.

41. Form the equation of the straight line parallel (perpendicular) to the straight line

$$ax + by + c = 0,$$

passing through the point of intersection of the straight lines

$$a_1x + b_1y + c_1 = 0 \quad \text{and} \quad a_2x + b_2y + c_2 = 0.$$

42. Find the conditions under which points (x_1, y_1) and $(x_2\ y_2)$ are positioned symmetrically about the straight line

$$ax + by + c = 0.$$

43. Form the equation of the curve $x^2 - y^2 = a^2$, with the straight lines

$$x + y = 0 \quad \text{and} \quad x - y = 0$$

as the coordinate axes.

Chapter IV

CONIC SECTIONS

1. Polar Coordinates

We draw a ray g from an arbitrary point in a plane, and fix a direction in which an angle is measured about O. Then the position of any point A in the plane may be specified by an ordered pair (ρ, θ): (1) ρ is the distance of the point A from O, and (2) θ is the angle between the ray OA and the ray g, (Fig. 31).

The numbers (ρ, θ) are called the *polar coordinates of the point A*. The fixed reference point O is called the pole, and the ray g beginning at O is called the *polar axis*.

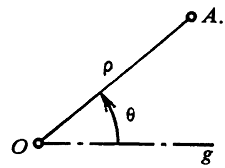

Fig. 31

As in the case of Cartesian coordinates, here we may speak of the equation of a curve in polar coordinates. Namely, the equation

$$\varphi(\rho, \theta) = 0$$

is called the *equation of a curve in polar coordinates* if the polar coordinates of each point of the curve satisfy this equation. And conversely, any ordered pair (ρ, θ) which satisfies this equation represents the polar coordinates of one of the points on the curve.

By way of example let us write an equation in polar coordinates for a circle passing through the pole, with centre on the polar axis

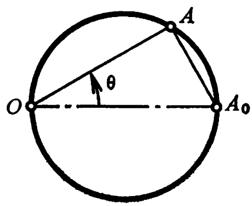

Fig. 32

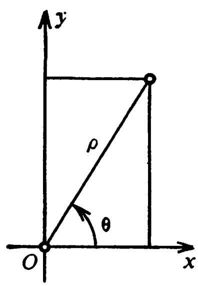

Fig. 33

and radius R. From a right-angled triangle OAA_0 we get $OA = OA_0 \times \cos\theta$ (Fig. 32). Whence the equation of the circle is

$$\rho = 2R\cos\theta.$$

Let us now introduce on the plane $\rho\theta$ an xy-coordinate system, taking the pole O as the origin of the Cartesian coordinate system and the polar axis as the positive semi-axis x, and choose the direction of the positive semi-axis y so that in the chosen direction it forms an angle of $+\pi/2$ with the polar axis.

The following simple relationship is obviously established between polar and rectangular coordinates of a point:

$$x = \rho \cos \theta, \quad y = \rho \sin \theta \qquad (*)$$

(Fig. 33).

We can get the equation of a curve in Cartesian coordinates, given the equation of the curve in polar coordinates, and vice versa.

Let us, for instance, form an equation of an arbitrary straight line in polar coordinates. The equation of a straight line in Cartesian coordinates is

$$ax + by + c = 0.$$

Introducing the ordered pair (ρ, θ) in this equation (instead of (x, y)) according to the formulas $(*)$, we get

$$\rho (a \cos \theta + b \sin \theta) + c = 0,$$

Assuming further

$$\frac{a}{\sqrt{a^2+b^2}} = \cos \alpha, \quad \frac{b}{\sqrt{a^2+b^2}} = \sin \alpha, \text{ and } \quad \frac{c}{\sqrt{a^2+b^2}} = -\rho_0,$$

we obtain the equation of the straight line in the form

$$\rho \cos (\alpha - \theta) = \rho_0.$$

2. Conic Sections

A *conic section* (or a conic) is a curve in which a plane, not passing through the cone's vertex, intersects a cone (Fig. 34). Conics possess a number of remarkable properties, one of them consisting in the following.

Each conic section, except for a circle, is a plane locus of points the ratio of whose distances from a fixed point F and a fixed line δ is constant. The point F is called the *focus* of a conic, the line δ its *directrix*.

Let us prove this property. Let γ be the curve in which the plane σ intersects a cone (Fig. 35). We now inscribe a sphere in the cone, which touches the plane σ and denote by F the point of contact of the sphere with the plane. Let ω be the plane containing the circle along which the sphere touches the cone. We then take an arbitrary point M on the curve γ and draw through it a generator of the cone, and denote by B the point of its intersection with the plane ω. We then drop a perpendicular from the point M to the line δ of intersection of the planes σ and ω.

The curve γ is said to possess the above property with respect to the point F and the line δ. Indeed, FM equals BM as tangents to the sphere drawn from one point. Further, if we denote by h the distance

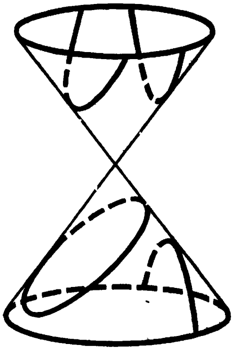

Fig. 34

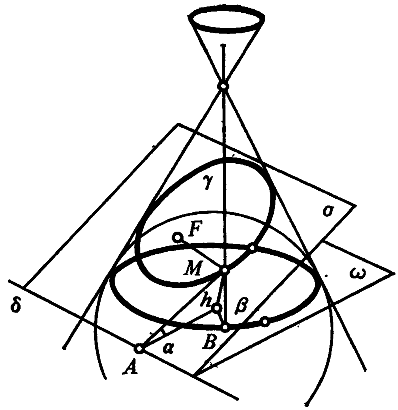

Fig. 35

of M from the plane ω, then $AM = h/\sin\alpha$, $BM = h/\sin\beta$, where α is the angle between the planes ω and σ and β is the angle between the generator of the cone and the plane ω.

Hence it follows that

$$\frac{AM}{FM}=\frac{AM}{BM}=\frac{\sin\beta}{\sin\alpha},$$

i.e. the ratio AM/FM does not depend on the point M. The statement has been proved.

Depending on the ratio λ of the distances of an arbitrary point of a conic from the focus and the directrix, the curve is an *ellipse* ($\lambda < 1$), a *parabola* ($\lambda = 1$), or a *hyperbola* ($\lambda > 1$). The number λ is called the *eccentricity* of the conic section.

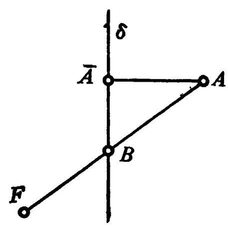

Fig. 36

Let F be the focus of a conic section and δ its directrix (Fig. 36). In case of an ellipse and a parabola ($\lambda \leqslant 1$) all points of the curve are on the one side of the directrix, namely, on the side where the focus F is located. Indeed, for any point A lying on the other side of the directrix

$$\frac{AF}{A\bar{A}} > \frac{AB}{A\bar{A}} \geqslant 1.$$

On the contrary, the hyperbola ($\lambda > 1$) has points located on both sides of the directrix. The hyperbola consists of two branches separated by the directrix.

3. Equations of Conic Sections in Polar Coordinates

Let us form the equation of a conic section (a conic) in polar coordinates $\rho\theta$ with the focus of the conic as the pole, and the polar axis drawn so that it is perpendicular to the directrix and intersects the latter (Fig. 37).

Let p be the distance from the focus to the directrix. The distance from an arbitrary point A of the conic to the focus is ρ and the dis-

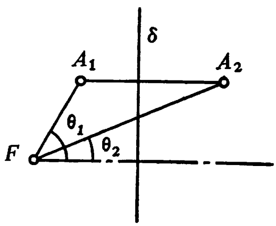

Fig. 37

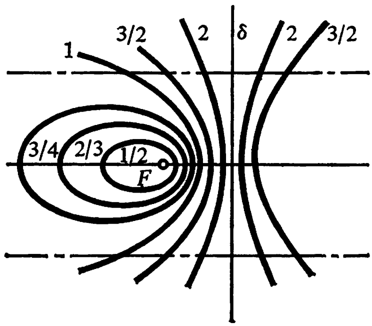

Fig. 38

tance from the directrix is $p-\rho\cos\theta$ or $\rho\cos\theta-p$, depending on whether A and F lie on one or on opposite sides of the directrix. Hence the equation of the conic section is

$$\frac{\rho}{p-\rho\cos\theta}=\lambda \qquad (*)$$

for the ellipse and parabola, and

$$\frac{\rho}{p-\rho\cos\theta}=\pm\lambda \qquad (**)$$

for the hyperbola (the plus sign corresponds to one branch of the hyperbola, and the minus to the other).

Solving equations (*) and (**) for ρ gives

$$\rho=\frac{\lambda p}{1+\lambda\cos\theta},$$

i.e. the *equation of the ellipse and parabola*, and

$$\rho=\frac{\pm\lambda p}{1\pm\lambda\cos\theta},$$

i.e. the *equation of the hyperbola*. The plus sign corresponds to one branch of the hyperbola, the minus to the other.

Figure 38 illustrates the dependence of the type of the conic section on the eccentricity λ.

4. Canonical Equations of Conic Sections in Rectangular Cartesian Coordinates

In Sec. 3 we obtained the equations of conic sections in polar coordinates $\rho\theta$. Let us now pass over to rectangular coordinates x, y and take the pole O as the origin and the polar axis as the positive semi-axis x.

From equations (*) and (**) of Sec. 3 for any conic section, we have

$$\rho^2 = \lambda^2 (p - \rho \cos \theta)^2.$$

Whence, taking into account formulas of Sec. 1 which establish the relationship between the polar and the Cartesian coordinates of a point, we obtain

$$x^2 + y^2 = \lambda^2 (p - x)^2,$$

or

$$(1 - \lambda^2)x^2 + 2p\lambda^2 x + y^2 - \lambda^2 p^2 = 0. \qquad (*)$$

This equation becomes much more simple, if we displace the origin along the x-axis accordingly.

Let us begin with an ellipse and a hyperbola. In this case equation (*) may be written in the following way:

$$(1-\lambda^2)\left(x+\frac{p\lambda^2}{1-\lambda^2}\right)^2+y^2-\frac{p^2\lambda^2}{1-\lambda^2}=0.$$

We now introduce the new coordinates x', y', using the formulas

$$x+\frac{\lambda^2 p}{1-\lambda^2}=x', \quad y=y',$$

which corresponds to the transfer of the origin into the point

$$\left(-\frac{\lambda^2 p}{1-\lambda^2}, 0\right).$$

Then the equation of a curve will take the form

$$(1-\lambda^2)\, x'^2+y'^2-\frac{\lambda^2 p^2}{1-\lambda^2}=0,$$

or, by putting for brevity

$$\frac{\lambda^2 p^2}{(1-\lambda^2)^2}=a^2, \qquad \frac{\lambda^2 p^2}{|1-\lambda^2|}=b^2,$$

we get the following equations:

for the ellipse

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1=0,$$

for the hyperbola

$$\frac{x'^2}{a^2}-\frac{y'^2}{b^2}-1=0.$$

The parameters a and b are termed the semi-axes of an ellipse (a hyperbola).

For the parabola ($\lambda = 1$) the equation (*) will have the form

$$2px + y^2 - p^2 = 0,$$

or

$$y^2-2p\left(-x+\frac{p}{2}\right)=0.$$

Introducing new coordinates

$$x' = -x+\frac{p}{2}, \quad y'=y$$

we obtain the equation of the form

$$y'^2 - 2px' = 0.$$

The equations of conic sections, obtained in the coordinates x', y' are called *canonical.*

Form the equation of the conic section with focus F (x_0, y_0), *directrix* $ax + by + c = 0$ *and eccentricity* λ.

Solution. We reduce the equation of the directrix to normal form. We will obtain

$$\frac{ax+by+c}{\sqrt{a^2+b^2}}=0.$$

The distance from the point (x, y) to the focus is

$$\sqrt{(x-x_0)^2+(y-y_0)^2}.$$

The distance from this point to the directrix is

$$\frac{|ax+by+c|}{\sqrt{a^2+b^2}}.$$

Since the ratio of these distances is equal to the eccentricity λ then the equation of the conic section will be

$$\frac{\sqrt{a^2+b^2}\sqrt{(x-x_0)^2+(y-y_0)^2}}{|ax+by+c|}=\lambda$$

or

$$(x-x_0)^2+(y-y_0)^2=\frac{\lambda^2}{a^2+b^2}(ax+by+c)^2.$$

5. Types of Conic Sections

Ellipse (Fig. 39):

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

We should note here that the axes of coordinates are the axes of symmetry of an ellipse, and the origin is the centre of symmetry. Indeed, if the point (x, y) belongs to an ellipse, then the points sym-

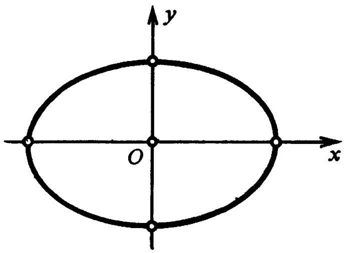

Fig. 39

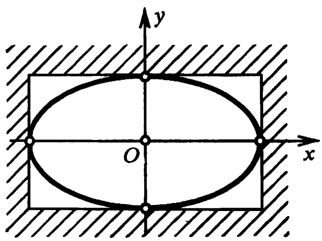

Fig. 40

metric to it about the coordinate axes $(-x, y)$, $(x, -y)$ and about the origin $(-x, -y)$ also belong to the ellipse, since they as well as the point (x, y) satisfy the equation of an ellipse. The points of intersection of an ellipse with its axes of symmetry are called *vertices of an ellipse.*

The entire ellipse is inside a rectangle $|x| \leqslant a$, $|y| \leqslant b$ formed by the tangents to the ellipse at its vertices (Fig. 40).

Indeed, if the point (x, y) is outside the rectangle, then at least one of the inequalities $|x| > a$ or $|y| > b$ is satisfied for it, but then

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}>1$$

and the point cannot belong to the ellipse.

We can obtain an ellipse from a circle by uniformly contracting the latter. Let us draw a circle on the plane

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}=1. \qquad (*)$$

We then imagine that the xy-plane is uniformly contracted with respect to the x-axis so that the point (x, y) is moved to the point (x', y'), where $x' = x$, and $y' = \frac{b}{a}y$. Then the circle (*) is transformed into a curve (Fig. 41). The coordinates of any of its points satisfy the equation

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1.$$

Hence, this curve is an ellipse.

Hyperbola (Fig. 42):

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Just as in the case of an ellipse, we conclude that the *axes of coordinates are the axes of symmetry of a hyperbola, and the origin is the centre of symmetry.*

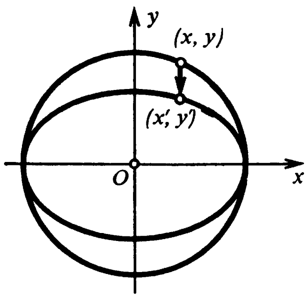

Fig. 41

The hyperbola consists of two branches symmetric about the y-axis and lying outside the rectangle $|x|<a$, $|y|<b$ and inside the two angles formed by its extended diagonals (Fig. 43).

Actually inside the rectangle $|x|<a$ and, consequently,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}<1,$$

i.e. there are no points of the hyperbola inside the rectangle. There are no such points within the hatched portion of the plane either (see Fig. 43), since for any point (x, y) located in this portion of the plane

$$\frac{b}{a}<\frac{|y|}{|x|},$$

whence

$$\frac{|x|}{a}<\frac{|y|}{b}$$

and, consequently,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}<0<1.$$

It is worth mentioning another property of a hyperbola. If a point (x, y), moving along the hyperbola goes away from the origin of the coordinates ($x^2+y^2\to\infty$), then its distance from one of the diagonals of the rectangle, which are obviously specified by the equations

$$\frac{x}{a}+\frac{y}{b}=0, \quad \frac{x}{a}-\frac{y}{b}=0,$$

decreases infinitely (tends to zero).

The straight lines

$$\frac{x}{a}+\frac{y}{b}=0, \quad \frac{x}{a}-\frac{y}{b}=0$$

are called the *asymptotes to the hyperbola.*

The hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$$

is said to be *conjugate* with respect to the considered hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

The conjugate hyperbola has the same asymptotes but is situated inside the auxiliary vertical angles formed by the asymptotes (Fig. 44).

Parabola (Fig. 45):

$$y^2-2px=0$$

has the x-axis as the *axis of symmetry*, since along with the point (x, y) a point $(x, -y)$ which is symmetric to it about the x-axis also

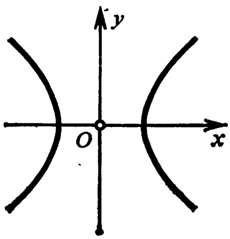

Fig. 42

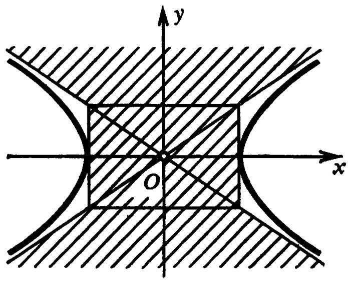

Fig. 43

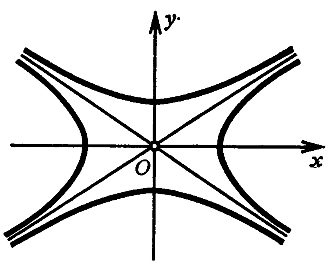

Fig. 44

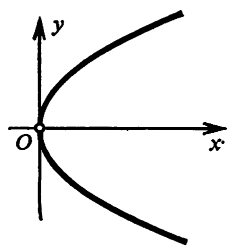

Fig. 45

belongs to the curve. The point of intersection of the parabola with its axis is called the *vertex of the parabola*. Thus, in this case the origin is the vertex of the parabola.

6. Tangent Line to a Conic Section

The *tangent line* to a curve at point A is, by definition, the limiting position of the secant AB as the point B draws nearer and nearer to the point A (Fig. 46).

Let a curve be given by the equation $y = f(x)$. Let us form the equation for a tangent line at point $A(x_0, y_0)$. Let $B(x_0 + \Delta x, y_0 + \Delta y)$ be a point of the curve situated close to A. The equation for the secant line is

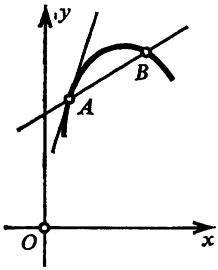

Fig. 46

$$y - y_0 = \frac{\Delta y}{\Delta x}(x - x_0).$$

As $B \to A$

$$\frac{\Delta y}{\Delta x} \to f'(x_0),$$

and we get the *equation for the tangent line*

$$y - y_0 = f'(x_0)(x - x_0). \qquad (*)$$

Similarly, if a curve is specified by the equation $x = \varphi(y)$, then the equation of the tangent line at point (x_0, y_0) will be

$$x - x_0 = \varphi'(y_0)(y - y_0). \qquad (**)$$

Let us form an equation of a tangent line to a conic section.

The case of the parabola. The equation of a parabola may be written in the form

$$x = \frac{y^2}{2p}.$$

Then the equation of a tangent line in the form (**) will be

$$x - x_0 = \frac{y_0}{p}(y - y_0)$$

or

$$yy_0 - y_0^2 + px_0 - px = 0.$$

Since the point (x_0, y_0) lies on the parabola and, hence, $y_0^2 - 2px_0 = 0$, the equation of the tangent line can be represented in the following final form:

$$yy_0 - p(x + x_0) = 0.$$

The case of the ellipse (hyperbola). Let (x_0, y_0) be a point on the ellipse, and $y_0 \neq 0$. In the vicinity of this point an ellipse can be specified by the equation

$$y = b\sqrt{1 - \frac{x^2}{a^2}},$$

where the square root should be taken with the same sign as y_0. The equation of a tangent line is found by the formula (*):

$$y-y_0=-\frac{x_0 b}{a^2\sqrt{1-\frac{x_0^2}{a^2}}}(x-x_0),$$

or

$$y-y_0=-\frac{x_0 b^2}{y_0 a^2}(x-x_0).$$

Multiplying it by y_0/b^2 and transposing all terms to the left-hand side, we get

$$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}-\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}\right)=0,$$

or

$$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}-1=0,$$

since $\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1$.

In the vicinity of any point (x_0, y_0) of ellipse, where $x_0 \neq 0$ the ellipse can be specified by the equation

$$x=a\sqrt{1-\frac{y^2}{b^2}}.$$

The square root is taken with the same sign as x_0. Then, reasoning in a similar way and using formula (**) we get an equation for a tangent line

$$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1.$$

Since at each point of the ellipse x_0 and y_0 cannot both be equal to zero, then at any point (x_0, y_0) the equation of the tangent line to the ellipse will be

$$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1.$$

The equation of the tangent line to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

is obtained analogously and has the form

$$\frac{xx_0}{a^2}-\frac{yy_0}{b^2}=1.$$

Let us show that a *tangent line to a conic section has only one point in common with this section* (i.e. the *point of tangency*). Indeed, let us take, for example, an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The equation of the tangent line at point (x_0, y_0) is

$$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1.$$

We shall now look for the points of intersection of the ellipse with its tangent line. Eliminating x from the equations, we obtain for y

$$\frac{y^2}{b^2}+\frac{a^2}{x_0^2}\left(\frac{yy_0}{b^2}-1\right)^2-1=0,$$

or

$$y^2\,\frac{a^2}{b^2x_0^2}\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}\right)-2y\,\frac{a^2y_0}{x_0^2b^2}+\frac{a^2}{x_0^2}\left(1-\frac{x_0^2}{a^2}\right)=0.$$

Since the point (x_0, y_0) lies on the ellipse, we have $x_0^2/a^2+y_0^2/b^2=1$, and the equation for y has the form

$$\frac{a^2}{b^2x_0^2}\,(y^2-2yy_0+y_0^2)=0.$$

This equation has two coinciding roots $y=y_0$. Similarly, eliminating y from the equations of the ellipse and its tangent line, we get $x=x_0$. Thus, the ellipse has only one point in common with the tangent line, i.e. the point of tangency (x_0, y_0). For the hyperbola and parabola this is proved in a similar way.

Find the equation of tangents to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

parallel to the straight line $y=kx$.

Solution. Any straight line parallel to the given one is described by the equation of the form $y=kx+c$. We will look for the points of intersection of this line with the ellipse. Substituting $y=kx+c$ into the equation of the ellipse gives

$$x^2\left(\frac{1}{a^2}+\frac{k^2}{b^2}\right)+2x\frac{kc}{b^2}+\frac{c^2}{b^2}-1=0. \qquad (***)$$

Among the straight lines $y=kx+c$ the tangents differ in that they have only one point of intersection with the ellipse. This means that the quadratic equation (***) has merged roots. And in that case the discriminant of the equation is zero, i.e.

$$\left(\frac{1}{a^2}+\frac{k^2}{b^2}\right)\left(\frac{c^2}{b^2}-1\right)-\frac{k^2c^2}{b^4}=0.$$

From this we find the values of c for which the straight line $y=kx+c$ will be tangent to the ellipse.

7. Focal Properties of Conic Sections

By definition, a conic section has a focus and a directrix. We are going to show that the *ellipse and hyperbola have one more focus and one more directrix*. Indeed, let the conic section be an ellipse. In the canonical arrangement its directrix δ_1 is parallel to the y-axis and the focus F_1 lies on the x-axis (Fig. 47). The equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Since in such a position the ellipse is symmetric about the y-axis, it has a focus F_2 and a directrix δ_2 which are symmetric with respect

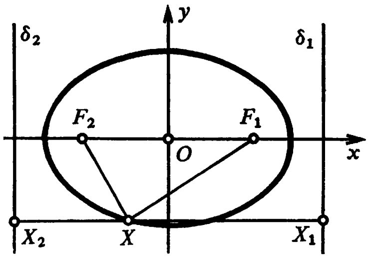

Fig. 47

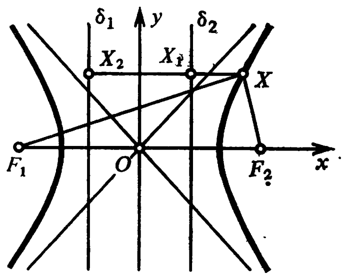

Fig. 48

to the focus F_1 and the directrix δ_1 about the y-axis. Reasoning similarly, we prove that the hyperbola also has two foci and two directrices.

We shall now show that the *sum of the distances from a point of the ellipse to its foci is constant*, i.e. independent of the point. Actually, for an arbitrary point X (Fig. 47) we have

$$\frac{XF_1}{XX_1}=\lambda, \qquad \frac{XF_2}{XX_2}=\lambda.$$

Hence

$$XF_1+XF_2=\lambda\,(X_1X_2)=\text{const}.$$

We can also show that the *difference between the distances of an arbitrary point of the hyperbola and its foci is constant* (Fig. 48).

Let us find the foci of the ellipse and hyperbola in canonical representation.

The equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Let c be the distance from the centre of the ellipse to the foci. The sum of the distances from the vertex $(0, b)$ to the foci is equal to $2\sqrt{b^2+c^2}$. The sum of the distances from the vertex $(a, 0)$ to the foci is equal to $2a$. Hence

$$\sqrt{b^2+c^2}=a,$$

and, consequently,

$$c=\sqrt{a^2-b^2}.$$

The equation of the hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

We then compare the difference between the distances from the point on the hyperbola with the abscissa c (where c is the distance from the centre of the hyperbola to the foci) with the difference between the distances from the vertex $(a, 0)$ to the foci. This comparison yields the following formula for the distance c

$$c=\sqrt{a^2+b^2}.$$

We must note the following reflection property of the ellipse: *A light ray emanating from one focus and reflected by the ellipse will pass through the other focus.* In other words, if $A(x_0, y_0)$ is a point on the ellipse, then the line segments AF_1 and AF_2 make equal angles with the tangent line at the point A.

To prove this, it is sufficient to show that the ratio of the distances from the focus to the tangent line and to the point of tangency A does not depend on the focus taken: F_1 or F_2.

The square of the distance from the focus $F_1(c, 0)$ to the point of tangency $A(x_0, y_0)$ is

$$AF_1^2=(x_0-c)^2+y_0^2=(x_0-c)^2+\left(b^2-\frac{x_0^2b^2}{a^2}\right)$$
$$=x_0^2\left(1-\frac{b^2}{a^2}\right)-2cx_0+b^2+c^2,$$

or, noting that $a^2=b^2+c^2$,

$$AF_1^2=\frac{x_0^2c^2}{a^2}-2cx_0+a^2=\left(\frac{cx_0}{a}-a\right)^2.$$

The distance from the focus $F_1(c, 0)$ to the tangent line at the point $A(x_0, y_0)$ is

$$h_1=k\left|\frac{cx_0}{a^2}-1\right|,$$

where k is a normalization factor reducing the equation of the tangent line to normal form.

Whence it follows that

$$\frac{h_1}{AF_1}=\frac{k}{a}.$$

For the other focus F_2 $(-c, 0)$ the same relation is obviously obtained. The assertion is thus proved.

The hyperbola possesses a similar optical property: *A light ray emanating from one focus and reflected by the hyperbola* (Fig. 49) *will seem to have come from the other focus.* The reflection property of the parabola consists in that light rays emanating from its focus become parallel to its axis on being reflected by the parabola.

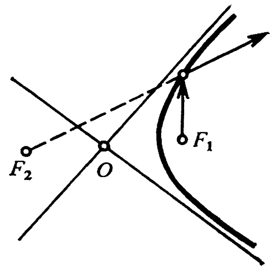

Fig. 49

8. Diameters of a Conic Section

The *diameter of an ellipse* (*a hyperbola*) is a line passing through the centre of the ellipse (hyperbola). The *diameter of a parabola* is a line parallel to its axis, and the axis itself.

An arbitrary line intersects a conic section at most at two points. If there are two points of intersection, then the line segment with the ends at the points of intersection is termed the *chord.* A conic section has the following property: *The midpoints of parallel chords lie on the diameter* (Fig. 50).

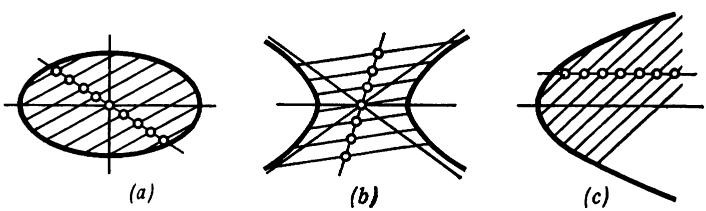

Fig. 50

This property is obvious if the chords are perpendicular to the axis of symmetry. In this case the midpoints of the chords lie on this axis.

Consider the general case. A family of parallel lines not parallel to the coordinate axes can be specified by the following equations

$$y = kx + b, \quad k \neq 0,$$

where k is the same for all lines.

The equations for the ellipse and hyperbola can be combined in the following way:

$$\alpha x^2 + \beta y^2 - 1 = 0.$$

The endpoints of the chords satisfy the system of simultaneous equations

$$\alpha x^2 + \beta y^2 - 1 = 0, \quad y = kx + b.$$

Substituting $kx + b$ for y in the first equation, we obtain the equation which is satisfied by the abscissas x_1 and x_2 of the endpoints of the chord:

$$(\alpha + \beta k^2) x^2 + 2\beta kbx + \beta b^2 - 1 = 0.$$

By the property of the roots of a quadratic equation

$$x_1 + x_2 = -\frac{2\beta kb}{\alpha + \beta k^2}.$$

Thus, the abscissa of the midpoint of the chord

$$x_c = \frac{x_1 + x_2}{2} = -\frac{\beta kb}{\alpha + \beta k^2}.$$

The ordinate y_c is found by substituting x_c in the equation of the chord $y = kx + b$:

$$y_c = -\frac{\beta k^2 b}{\alpha + \beta k^2} + b = \frac{\alpha b}{\alpha + \beta k^2}.$$

Whence

$$y_c = -\frac{\alpha}{\beta k} x_c.$$

Thus, the midpoints of parallel chords $y = kx + b$ lie on a straight line passing through the origin, i.e. through the centre of the ellipse (hyperbola). Its slope

$$k' = -\frac{\alpha}{\beta k}.$$

The diameter

$$y = k'x$$

is called *conjugate* to the diameter

$$y = kx,$$

which is parallel to the chords.

Obviously, the diameters are mutually conjugate, since the slope of the diameter conjugate to

$$y = k'x$$

is

$$-\frac{\alpha}{\beta k'} = k.$$

Let us consider the case of parabola. The coordinates of the endpoints of the chords satisfy the system

$$y^2 - 2px = 0, \quad y = kx + b.$$

Eliminating x, we find the equation for the ordinates of the end-points:

$$y^2 - \frac{2py}{k} + \frac{2pb}{k} = 0.$$

Hence, like the previous case

$$y_1 + y_2 = \frac{2p}{k}.$$

Thus,

$$y_c = \frac{y_1 + y_2}{2} = \frac{p}{k} = \text{const}.$$

The midpoints of the chords lie on a line parallel to the x-axis (the axis of the parabola).

Let us mention one more property of conjugate diameters: *If a diameter intersects a conic section, then the tangent lines at the points of intersection are parallel to the conjugate diameter.*

Actually, let (x_0, y_0) be the point of intersection of the diameter $y = kx$ with an ellipse (hyperbola) $\alpha x^2 + \beta y^2 = 1$. The equation for a tangent line at the point (x_0, y_0) is $\alpha x x_0 + \beta y y_0 - 1 = 0$. Its slope $k' = -\alpha x_0/\beta y_0$. Since the point (x_0, y_0) lies on the diameter $y = kx$, we have $y_0 = kx_0$. Therefore

$$k' = -\frac{\alpha}{\beta k},$$

which was required to prove.

Note that in the case of a circle, the diameter conjugate to the given one is the diameter perpendicular to it. This follows from a theorem in elementary geometry: the midpoints of parallel chords of a circle lie on the diameter perpendicular to the chords.

9. Curves of the Second Degree

A second-degree curve is the locus of points in the plane, whose coordinates satisfy an equation of the form

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a = 0, \qquad (*)$$

in which at least one of the coefficients a_{11}, a_{12}, a_{22} is non-zero.

This definition is, obviously, *invariant relative to a coordinate system*, since the coordinates of a point in any other coordinate system are expressed linearly in terms of its coordinates in the xy-system and, consequently, the equation in any other coordinate system will have the form (*).

Let us consider what is meant by second-degree curves.

We consider the curve in a new coordinate system $x'y'$ which is related to the xy-system by the formulas

$$x = x' \cos\alpha + y' \sin\alpha,$$
$$y = -x' \sin\alpha + y' \cos\alpha.$$

The equation for the curve of the form (*) will have the following coefficient of $x'y'$:

$$2a'_{12} = 2a_{11}\cos\alpha\sin\alpha - 2a_{22}\sin\alpha\cos\alpha + 2a_{12}(\cos^2\alpha - \sin^2\alpha)$$
$$= (a_{11} - a_{22})\sin 2\alpha + 2a_{12}\cos 2\alpha.$$

Obviously, it is always possible to choose an angle α so that this coefficient is equal to zero. Therefore, without loss of generality, we can assume that in the initial equation (*) $a_{12} = 0$.

We shall consider two cases:

Case A: both coefficients a_{11} and a_{22} are non-zero.

Case *B*: one of the coefficients, either a_{11} or a_{22}, is equal to zero. Without loss of generality, we shall consider $a_{11} = 0$.

In case *A*, using a new coordinate system $x'y'$,

$$x' = x + \frac{a_1}{a_{11}}, \quad y' = y + \frac{a_2}{a_{22}},$$

we bring the equation (*) to the form

$$a_{11}x'^2 + a_{22}y'^2 + c = 0 \qquad (**)$$

and introduce the following subcases:

A_1: $c \neq 0$, a_{11} and a_{22} are of the same sign which is opposite to the sign of c. The curve is obviously an ellipse.

A_2: $c \neq 0$, a_{11} and a_{22} have different signs. The curve is a hyperbola.

A_3: $c \neq 0$, a_{11}, a_{22} and c have the same sign. None of the real points satisfies the equation. The curve is called *imaginary*.

A_4: $c = 0$, a_{11} and a_{22} have different signs. The curve decomposes into two lines, since the equation (**) can be written in the form

$$\left(x' - \sqrt{-\frac{a_{22}}{a_{11}}}\, y'\right)\left(x' + \sqrt{-\frac{a_{22}}{a_{11}}}\, y'\right) = 0.$$

A_5: $c = 0$, a_{11} and a_{22} have the same sign. The equation can be written in the form

$$\left(x' - i\sqrt{\frac{a_{22}}{a_{11}}}\, y'\right)\left(x' + i\sqrt{\frac{a_{22}}{a_{11}}}\, y'\right) = 0.$$

The curve decomposes into a pair of imaginary lines intersecting at a real point (0, 0).

Let us now consider Case B.

In this case, by using the new coordinate system $x'y'$:

$$x' = x, \quad y' = y + \frac{a_2}{a_{22}},$$

we reduce the equation to the form

$$2a_1x' + a_{22}y'^2 + c = 0. \qquad (***)$$

We then distinguish the following subcases:

B_1: $a_1 \neq 0$. The curve is a parabola, since by transferring (or changing) to the new coordinates

$$x'' = x' + \frac{c}{2a_1}, \quad y'' = y'$$

we reduce the equation (***) to the form

$$2a_1x'' + a_{22}y''^2 = 0.$$

B_2: $a_1 = 0$, a_{22} and c have different signs. The curve decomposes into a pair of parallel straight lines

$$y \pm \sqrt{-\frac{c}{a_{22}}} = 0.$$

B_3: $a_1 = 0$, a_{22} and c are of the same sign. The curve decomposes into a pair of imaginary non-intersecting lines

$$y \pm i\sqrt{\frac{c}{a_{22}}} = 0.$$

B_4: $a_1 = 0$, $c = 0$. The curve is a pair of coinciding straight lines.

Thus, *a real curve of the second degree represents either a conic section (ellipse, hyperbola, parabola), or a pair of straight lines (which may even coincide).*

EXERCISES TO CHAPTER IV

1. Show that the equation for any circle in polar coordinates can be written in the form

$$\rho^2 + 2a\rho \cos(\alpha + \theta) + b = 0.$$

Determine the coordinates of its centre, ρ_0, θ_0, and the radius R.

2. Express the distance between two points in terms of polar coordinates of these points.

3. What geometric meaning have α and ρ_0 in the equation of a line in polar coordinates

$$\rho \cos(\alpha - \theta) = \rho_0?$$

4. Form an equation (in polar coordinates) of the locus of feet of perpendiculars dropped from the point A on the circle onto its tan-

gent lines (the *cardioid*, see Fig. 51). Take the point A as the pole, and the extension of radius OA as the polar axis.

5. Form the equation for the *lemniscate of Bernoulli* which is the name for the locus of points the product of whose distances from the two given points F_1 and F_2 (the foci) is constant and equal to $|F_1F_2|^2/4$. Take the midpoint of the line segment joining the foci as the pole, and the ray passing through one of the foci as the polar axis.

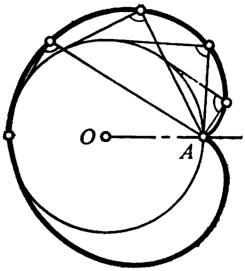

Fig. 51

6. Show that an intersection of a circular cylinder by a plane is an ellipse. What is the eccentricity of the ellipse if the plane intersects it at an acute angle α.

7. Show that the curve

$$\rho = \frac{c}{1 + a\cos\theta + b\sin\theta}$$

is a conic section. Under what condition is the curve an ellipse, a hyperbola, a parabola?

8. Form the equation of an ellipse by the three points $(\rho_1, 0)$, $(\rho_2, \pi/2)$ and (ρ_3, π), knowing that one of its foci is situated at the pole of the $\rho\theta$ coordinate system.

9. Let A and B be the points at which a conic section intersects a straight line passing through the focus F. Prove that

$$\frac{1}{AF} + \frac{1}{BF}$$

does not depend on the straight line.

10. Show that the inverse transformation of the parabola with respect to the focus transforms it into a cardioid (see Exercise 4).

11. Show that a straight line intersects a conic section at most at two points.

12. Let k be any conic section and F its focus. Show that the distance for an arbitrary point A of the conic section to the focus F is expressed linearly in terms of paired coordinates x, y, i.e.,

$$AF = \alpha x + \beta y + \gamma,$$

where α, β, γ are constants.

13. Show that the locus of points, the sum of whose distances from the two given points is constant, is an ellipse.

14. Show that the locus of points the difference of whose distances from the two given points is constant is a hyperbola.

15. What is the locus of the centres of circles touching the two given circles k_1 and k_2? Consider various cases of mutual positions of the circles k_1 and k_2, and also the case when one of the circles degenerates into a straight line.

16. Justify the following method of constructing an ellipse (Fig. 52). The sides CD and AC of a rectangle are divided into the same number of segments of equal length. The points of division are then joined to A and B. The points of intersection thus obtained lie on the ellipse with the major axis AB. The minor semi-axis is equal to half the altitude of the rectangle.

17. Justify the method of constructing the parabola illustrated in Fig. 53.

18. Express the distances from the point (x, y) of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to its asymptote in terms of the abscissa (x) of the point. Show that the distance from (x, y) to one of the asymptotes falls off indefinitely as $|x| \to \infty$.

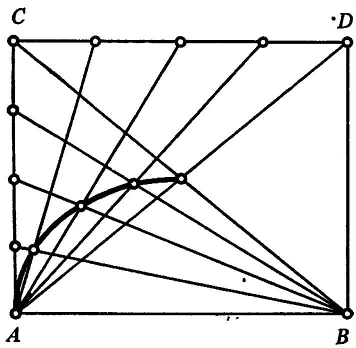

Fig. 52

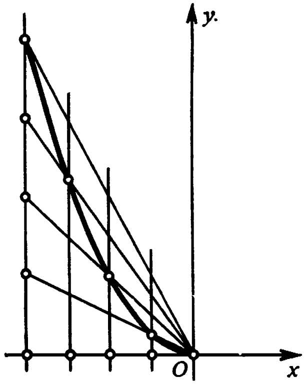

Fig. 53

19. Show that the product of the distances from a point on a hyperbola to its asymptote is constant, i.e. independent of the point.

20. Show that the orthogonal projection of a circle on a plane is an ellipse.

21. Show that a straight line parallel to the axis of a parabola intersects the parabola at one point.

22. Show that a straight line parallel to the asymptote of a hyperbola intersects the hyperbola at one point.

23. Show that the equation of a hyperbola with the asymptotes

$$a_1x + b_1y + c_1 = 0 \text{ and } a_2x + b_2y + c_2 = 0$$

can be written as

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = \text{const.}$$

24. The tangents to the ellipse

$$\frac{x^2}{a^2}+\frac{x^2}{b^2}=1$$

have the slope k. Find the points of tangency.

25. Show that the segment of a tangent to a hyperbola between the asymptotes is bisected by the point of tangency.

26. Show that a tangent to a hyperbola together with the asymptotes bounds a triangle of constant area.

27. Express the condition of tangency of a straight line

$$y-y_0=\lambda(x-x_0)$$

to an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Show that the locus of vertices (x_0, y_0) of right angles whose sides touch the ellipse is a circle.

28. Show that the vertices of right angles whose sides touch a parabola lie on the directrix, and a straight line joining the points of tangency passes through the focus.

29. Justify the following method of construction of foci of the ellipse. From the vertex on the semiminor axis circumscribe a circle of radius equal to the semimajor axis. Then the points of intersection of this circle with the major axis will be the foci of the ellipse.

30. Prove the reflection property of the hyperbola.

31. Find the focus of the parabola in canonical representation.

32. Find the directrices of the conic sections in canonical representation.

33. Show that all conic sections k_λ given by the equations

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,$$

where λ is the parameter of the family, are confocal, i.e. have their foci coincident.

34. Show that through any point of the xy-plane not belonging to the coordinate axes pass two conic sections of the family k_λ (Exercise 33): an ellipse and a hyperbola.

35. Show that the ellipse and the hyperbola of the family k_λ (Exercise 33) which pass through the point (x_0, y_0) intersect at this point at right angles, i.e. the tangent lines to them at the point (x_0, y_0) are perpendicular.

36. The chord of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is bisected at the point (x_0, y_0). Find the slope of the chord.

37. Show that the ellipse allows a parametric representation:

$$x = a \cos t, \quad y = b \sin t.$$

What condition do the values of the parameter t corresponding to the end-points of conjugate diameters satisfy? Prove that the sum of the squares of conjugate diameters of the ellipse is constant (Apollonius theorem). Formulate and prove a similar theorem for the hyperbola.

38. Any ellipse can be represented as the projection of a circle. Show that perpendicular diameters of the circle correspond in this projection to conjugate diameters of the ellipse. Relying on this fact, prove that the area of the parallelogram formed by the tangent lines at the end-points of the conjugate diameters is constant.

39. Show that the area of any parallelogram with the vertices at the end-points of the conjugate diameters of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

has one and the same value equal to $2ab$.

40. It is known that of all the quadrilaterals inscribed in a circle the square has the greatest area. Show that among all the quadrilaterals inscribed in the ellipse the parallelograms with the vertices at the end-points of the conjugate diameters have the greatest area.

41. Show that the area of the ellipse with the semi-axes a and b is equal to πab.

42. Is it possible to inscribe a triangle in an ellipse so that the tangent line at each of its vertices is parallel to the opposite side? With what arbitrariness can it be done? What is the area of this triangle if the semi-axes of the ellipse are a and b.

43. Think, what are the curves given by the following equations:

(a) $x^2 - xy + y^2 - x + y = 0$,

(b) $xy + y^2 - x + y = 0$,

(c) $x^2 - 4xy + 4y^2 + x = 0$,

(d) $x^2 - y^2 + x + y = 0$,

(e) $x^2 + 2xy + y^2 - 1 = 0$.

44. Show that the second-degree curve

$$(ax + by + c)^2 - (a_1x + b_1y + c_1)^2 = 0$$

decomposes into a pair of lines, possibly coincident ones.

45. As is known, all points of the ellipse are within a bounded portion of the xy-plane. Proceeding from this fact, show that the second-degree curve $(ax + by + c)^2 + (\alpha x + \beta y + \gamma)^2 = k^2$ is an ellipse if the expressions $ax + by$ and $\alpha x + \beta y$ are independent and $k \neq 0$.

46. Show that the second-degree curve

$$(ax + by + c)(\alpha x + \beta y + \gamma) = k \neq 0$$

is a hyperbola, provided the expressions $ax + by$, $\alpha x + \beta y$ are independent.

47. Show that the second-degree curve

$$(ax + by + c)^2 - (\alpha x + \beta y + \gamma)^2 = k \neq 0.$$

is a hyperbola if $ax + by$, $\alpha x + \beta y$ are independent.

48. Show that if a line intersects a second-degree curve at three points, then the curve decomposes into a pair of lines, possibly coincident ones.

49. Show that if two indecomposable curves of the second degree have five points in common, then they coincide.

Chapter V

RECTANGULAR CARTESIAN COORDINATES AND VECTORS IN SPACE

1. Cartesian Coordinates in Space. Introduction

Let us take three mutually perpendicular straight lines x, y, z intersecting at one point O (Fig. 54). We then draw a plane through each pair of these straight lines. The plane through x and y is called the xy-plane. Two other planes are called the xz- and yz-planes, respectively. The straight lines x, y, z are called the *coordinate axes*, the point of their intersection O is called the *origin of coordinates* and the xy-, yz-, xz-planes are called the *coordinate planes*. The point O divides each coordinate axis into two half-lines. One of them is conventionally called positive, the other negative.

Now we take an arbitrary point A and draw through it a plane parallel to the yz-plane (Fig. 55). It will intersect the x-axis at a certain point A_x. The coordinate x of A will be the number whose absolute value is equal to the length of OA_x, and is positive if A_x lies to the right of the origin and negative if it lies to the left of the origin. If A_x coincides with the point O, then we take $x = 0$. Likewise, we find the coordinates y and z of A. We will write the coordinates of the point in parentheses after the symbol of the point, e.g. $A(x, y, z)$. Sometimes we will simply denote a point by its coordinates (x, y, z).

We now express the distance between two points A_1 (x_1, y_1, z_1) *and* A_2 (x_2, y_2, z_2) *in terms of the coordinates of these points.*

To begin with, we consider the case where the straight line A_1A_2 is not parallel to the z-axis (Fig. 56). We draw through A_1 and A_2 straight lines parallel to the z-axis. They will intersect the xy-plane at points $\bar{A}_1$ and $\bar{A}_2$. These points have the same coordinates x and

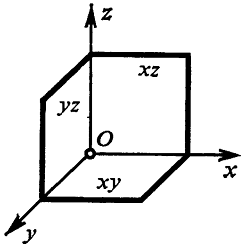

Fig. 54

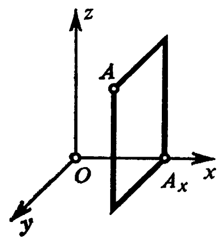

Fig. 55

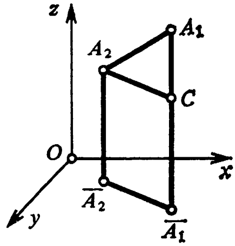

Fig. 56

y as A_1 and A_2, and their coordinate z is zero. We now draw a plane through A_2 parallel to the xy-plane. It will intersect the straight line $A_1\bar{A}_1$ at a certain point C. By the Pythagoras theorem

$$A_1A_2^2 = A_1C^2 + CA_2^2.$$

The segments CA_2 and $\bar{A}_1\bar{A}_2$ are equal, but

$$\bar{A}_1\bar{A}_2^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

The length of A_1C is $|z_1 - z_2|$. Therefore,

$$A_1A_2^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

If A_1A_2 is parallel to the z-axis, then $A_1A_2 = |z_1 - z_2|$. The same result is obtained using the formula just derived, since in that case $x_1 = x_2$, $y_1 = y_2$.

Let A (x_1, y_1, z_1) and B (x_2, y_2, z_2) be two arbitrary points. *We will express the coordinates* x, y, z *of point* C *that divides* AB *in the ratio* $\lambda : \mu$ *in terms of the coordinates of* A *and* B. To this end, we draw through A, B, C straight lines parallel to the z-axis. They will intersect the xy-plane at points A' $(x_1, y_1, 0)$, B' $(x_2, y_2, 0)$ and C' $(x, y, 0)$. By the property of parallel projection

$$\frac{A'C'}{C'B'} = \frac{AC}{CB} = \frac{\lambda}{\mu}.$$

As we know, in the xy-plane the coordinates of point C' that divides $A'B'$ in the ratio $\lambda : \mu$ are expressed as

$$x = \frac{\mu x_1 + \lambda x_2}{\lambda + \mu}, \qquad y = \frac{\mu y_1 + \lambda y_2}{\lambda + \mu}.$$

Similarly, if we project A, B, C on the xz-plane, we will have

$$x=\frac{\mu x_1+\lambda x_2}{\lambda+\mu}, \qquad z=\frac{\mu z_1+\lambda z_2}{\lambda+\mu}.$$

Thus, the point C has the coordinates

$$x=\frac{\mu x_1+\lambda x_2}{\lambda+\mu}, \qquad y=\frac{\mu y_1+\lambda y_2}{\lambda+\mu}, \qquad z=\frac{\mu z_1+\lambda z_2}{\lambda+\mu}.$$

2. Translation in Space

Translation in space is defined as a change such that an arbitrary point (x, y, z) of a solid is sent into the point $(x + a, y + b, z + c)$, where a, b, c are constants. Translation in space is given by the formulas

$$x' = x + a, \quad y' = y + b, \quad z' = z + c,$$

which express the coordinates x', y', z' of the point into which (x, y, z) is sent under the translation. As in a plane, the following properties of translation can be proved:

1. Translation is motion.

2. Under translation points move along parallel (or coincident) straight lines by the same distance.

3. Under translation each straight line is moved to a new parallel line (or to itself).

4. Whatever points A and A', there exists only one translation under which point A changes into A'.

5. Two consecutive translations yield a translation.

6. The transformation inverse of a translation is a translation.

In space translation acquires the following new property:

7. *Under translation in space each plane is moved either to itself or to a new parallel plane.*

Proof. Let α be an arbitrary plane. In this plane we draw two nonintersecting straight lines a and b. Under translation a and b change either into themselves or into parallel lines a' and b'. A plane α changes into a certain plane α' that passes through straight lines a' and b'. If α' does not coincide with α, then it is known to be parallel to α.

The *angle between skew lines* is the angle between the intersecting lines that are parallel to them. It follows from the properties of translation that the angle between skew lines is the same whichever parallel lines are taken.

The *angle between a straight line and a plane* is the angle between this line and its orthogonal projection on the plane, if the line is not perpendicular to the plane. If the line is perpendicular to the plane, the angle between them is considered to be 90°.

The *angle between intersecting planes* is taken to be equal to that between the straight lines obtained when these planes meet with the plane perpendicular to their intersection line. The angle between parallel planes is taken to be zero.

It follows from the properties of translation that the angle between the planes defined in this way is independent of the choice of the secant plane.

3. Vectors in Space

In space, as well as in the plane, a vector is a directed line segment. For vectors in space the same basic concepts are defined: magnitude, direction, equality of vectors.

The coordinates of a vector with origin at A_1 (x_1, y_1, z_1) and end at A_2 (x_2, y_2, z_2) are the numbers $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$. Just as for vectors in the plane, it is shown that equal vectors have equal respective coordinates, and conversely, vectors with equal respective coordinates are equal. This justifies the notation of vectors by their coordinates, e.g. **a** (a_1, a_2, a_3) or $\overrightarrow{(a_1, a_2, a_3)}$.

We define addition and scalar multiplication of vectors exactly as for vectors in the plane.

The sum of the vectors **a** (a_1, a_2, a_3) and **b** (b_1, b_2, b_3) is the vector **c** $(a_1 + b_1, a_2 + b_2, a_3 + b_3)$. And just as in the plane it is shown that vector addition in space obeys the commutative and associative laws. This means that for any two vectors **a** and **b** we have

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \text{ (commutative law)}$$

for any three vectors **a**, **b**, **c** we have

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c} \text{ (associative law).}$$

And as in the plane, we can prove the equality of vectors

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$

The product of a vector **a** (a_1, a_2, a_3) and a number λ is a vector **a**$'$ $(\lambda a_1, \lambda a_2, \lambda a_3)$. Just as in the plane, we can prove that the magnitude of $\lambda\mathbf{a}$ is $|\lambda|\,|\mathbf{a}|$, and its direction coincides with that of **a**, if $\lambda > 0$, and is opposite to that of **a**, if $\lambda < 0$.

Just as in the plane, we can prove that multiplication of a vector by a number shows two distributive properties, i.e. for any two vectors **a** and **b** and a number λ we have

$$\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b},$$

and for any two numbers λ and μ and a vector **a** we have

$$(\lambda + \mu)\,\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}.$$

The scalar product of two vectors **a** (a_1, a_2, a_3) and **b** (b_1, b_2, b_3) is the number $a_1b_1 + a_2b_2 + a_3b_3$. Just as in the plane, we can prove that the scalar product of two vectors in space is the product of their absolute values and the cosine of the angle between the vectors.

Just as in the plane, we can prove that the scalar product of two vectors shows the distributive property, i.e. for any three vectors **a**, **b**, **c** we have

$$(\mathbf{a} + \mathbf{b})\,\mathbf{c} = \mathbf{ac} + \mathbf{bc}.$$

4. Decomposition of a Vector into Three Non-coplanar Vectors

Just as in a plane, two non-zero vectors in space are called collinear if they lie on the same straight line or on parallel lines. Just as in the plane, we can prove that if a vector **b** is collinear with a vector **a** or is a zero vector, then $\mathbf{b} = \lambda\mathbf{a}$, where λ is a number.

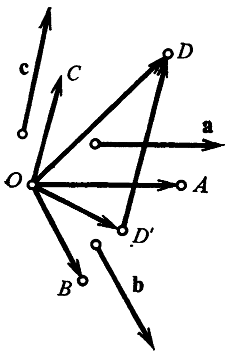

Fig. 57

Three non-zero vectors in space are called *coplanar*, if the vectors equal to them and having the same origin lie in one plane. Just as in the plane, any vector can be decomposed into two non-collinear vectors, *so in space any vector can be decomposed into three non-coplanar vectors in a unique manner*. We now prove this.

Let **a**, **b**, **c** be three non-coplanar vectors and **d** any vector. We now show that there is only one decomposition of **d**:

$$\mathbf{d} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}.$$

We draw from an arbitrary point O four vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ that are equal to **a**, **b**, **c** and **d**, respectively. We denote by α the plane in which $\overrightarrow{OA}$ and $\overrightarrow{OB}$ lie (Fig. 57). If point D lies on OC, then $\overrightarrow{OD} = \nu\overrightarrow{OC}$. Hence $\mathbf{d} = \nu\mathbf{c}$.

If point D does not lie on OC, then we draw through it a straight line parallel to OC. It will intersect the plane α at a certain point D'. The vectors $\overrightarrow{OC}$ and $\overrightarrow{D'D}$ are collinear. Therefore, $\overrightarrow{D'D} = \nu\overrightarrow{OC}$. The vector $\overrightarrow{OD'}$ lies in α, just as $\overrightarrow{OA}$ and $\overrightarrow{OB}$ do. Therefore, $\overrightarrow{OD'} = \lambda\overrightarrow{OA} + \mu\overrightarrow{OB}$. Since $\overrightarrow{OD} = \overrightarrow{OD'} + \overrightarrow{D'D}$, then

$$\overrightarrow{OD} = \lambda\overrightarrow{OA} + \mu\overrightarrow{OB} + \nu\overrightarrow{OC},$$

or

$$\mathbf{d} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}.$$

We have thus proved the existence of the decomposition of **d**.

Now prove its uniqueness. Suppose that there exists another decomposition

$$\mathbf{d} = \lambda'\mathbf{a} + \mu'\mathbf{b} + \nu'\mathbf{c}.$$

Then

$$(\lambda' - \lambda)\,\mathbf{a} + (\mu' - \mu)\,\mathbf{b} + (\nu' - \nu)\,\mathbf{c} = 0.$$

We multiply this equality in a scalar manner by a vector **e** that is perpendicular to **b** and **c**. Then

$$(\lambda' - \lambda)\,(\mathbf{ae}) = 0.$$

Since the vectors **a** and **e** are non-zero and not perpendicular, then $\mathbf{ae} \neq 0$, hence $\lambda' - \lambda = 0$. We can prove in a similar manner that $\mu' - \mu = 0$, $\nu' - \nu = 0$. This completes the proof of the uniqueness of the decomposition.

The unit vectors have the same direction as the coordinate axes and are denoted as $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, respectively, for the x-, y-, and z-axes. Then *for any vector* **a** (a_1, a_2, a_3) *we have the decomposition*

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3.$$

In fact,

$$\mathbf{a} = \overrightarrow{(a_1,\ 0,\ 0)} + \overrightarrow{(0,\ a_2,\ 0)} + \overrightarrow{(0,\ 0,\ a_3)}$$

$$= a_1\,\overrightarrow{(1,\ 0,\ 0)} + a_2\,\overrightarrow{(0,\ 1,\ 0)} + a_3\,\overrightarrow{(0,\ 0,\ 1)} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 = a_3\mathbf{e}_3.$$

5. Vector Product of Vectors

The vector product, or cross product, of the vectors **a** (a_1, a_2, a_3) and **b** (b_1, b_2, b_3) is defined as the vector **c** $(a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$. The vector product of **a** by **b** will be denoted by $\mathbf{a} \wedge \mathbf{b}$. It follows from the definition of vector product directly that $\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$. If one or both vectors are zero, then their vector product is a zero vector.

The vector product of collinear vectors is a zero vector. Conversely, if the vector product of non-zero vectors is a zero vector, then the vectors are collinear.

Proof. Let **a** (a_1, a_2, a_3) and **b** (b_1, b_2, b_3) be collinear vectors. Then $\mathbf{b} = \lambda\mathbf{a}$, and hence $b_1 = \lambda a_1$, $b_2 = \lambda a_2$, $b_3 = \lambda a_3$. Substituting these values of b_1, b_2, b_3 into the expression for $\mathbf{a} \wedge \mathbf{b}$, we see that all the coordinates of $\mathbf{a} \wedge \mathbf{b}$ are zero, and hence $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$.

We now prove the inverse statement. Let $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$. This means that $a_2b_3 - a_3b_2 = 0$, $a_3b_1 - a_1b_3 = 0$, $a_1b_2 - a_2b_1 = 0$. Hence

$$\frac{a_2}{b_2} = \frac{a_3}{b_3}, \quad \frac{a_3}{b_3} = \frac{a_1}{b_1}, \quad \frac{a_1}{b_1} = \frac{a_2}{b_2},$$

i.e. the coordinates of **a** and **b** are proportional, and so the vectors are collinear.

Let now **a** and **b** be non-zero and non-collinear vectors. We now find the direction and magnitude of the vector $\mathbf{a}\wedge\mathbf{b}$. We have

$$(\mathbf{a}\wedge\mathbf{b})\,\mathbf{a} = (a_2b_3 - a_3b_2)\,a_1 + (a_3b_1 - a_1b_3)\,a_2 + (a_1b_2 - a_2b_1)\,a_3 = 0.$$

Likewise, $(\mathbf{a}\wedge\mathbf{b})\,\mathbf{b} = 0$. The vector $\mathbf{a}\wedge\mathbf{b}$ is thus perpendicular to **a** and **b**.

We will find the magnitude of $\mathbf{a}\wedge\mathbf{b}$ using the identity

$$(a_2b_3 - a_3b_2)^2 + (a_3b_1 - b_3a_1)^2 + (a_1b_2 - a_2b_1)^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2.$$

We can verify this identity by direct check.

We note that $a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2$, $b_1^2 + b_2^2 + b_3^2 = |\mathbf{b}|^2$, $a_1b_1 + a_2b_2 + a_3b_3 = \mathbf{ab} = |\mathbf{a}|\ |\mathbf{b}| \cos\varphi$ (φ is the angle between **a** and **b**). Then

$$|\mathbf{a}\wedge\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\varphi.$$

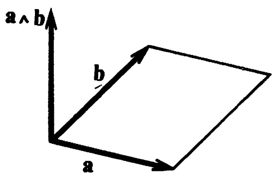

Fig. 58

We can assume without loss of generality that **a** and **b** have a common origin. In that case, $|\mathbf{a}|\ |\mathbf{b}|\sin\varphi$ is the area of the parallelogram, constructed on **a** and **b** (Fig. 58).

By way of exercise we find the area of the triangle with vertices at $A_1(x_1, y_1, z_1)$, $A_2(x_2, y_2, z_2)$, $A_3(x_3, y_3, z_3)$. The magnitude of $\overrightarrow{A_1A_2}\wedge\overrightarrow{A_1A_3}$ is the area of the parallelogram constructed on $\overrightarrow{A_1A_2}$ and $\overrightarrow{A_1A_3}$. The area of the parallelogram is twice the area of the triangle ABC. Thus,

$$S_\Delta = \frac{1}{2}\,|\overrightarrow{A_1A_2}\wedge\overrightarrow{A_1A_3}|.$$

The coordinates of $\overrightarrow{A_1A_2}\wedge\overrightarrow{A_1A_3}$ are

$$\begin{vmatrix} y_2-y_1 & z_2-z_1 \\ y_3-y_1 & z_3-z_1 \end{vmatrix},\quad \begin{vmatrix} z_2-z_1 & x_2-x_1 \\ z_3-z_1 & x_3-x_1 \end{vmatrix},\quad \begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{vmatrix}.$$

Therefore,

$$S_\Delta = \frac{1}{2}\left\{\begin{vmatrix} y_2-y_1 & z_2-z_1 \\ y_3-y_1 & z_3-z_1 \end{vmatrix}^2 + \begin{vmatrix} z_2-z_1 & x_2-x_1 \\ z_3-z_1 & x_3-x_1 \end{vmatrix}^2 + \begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{vmatrix}^2\right\}^{1/2}.$$

Specifically, if the triangle ABC lies in the xy-plane, then

$$S_\Delta = \frac{1}{2} \left\| \begin{matrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{matrix} \right\| .$$

6. Scalar Triple Product of Vectors

The scalar triple product of the vectors **a** (a_1, a_2, a_3), **b** (b_1, b_2, b_3) and **c** (c_1, c_2, c_3) taken in this order is defined as the number

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} .$$

The scalar triple product of vectors is denoted as (**abc**).

The name is better explained by the following expression:

$$(\mathbf{abc}) = \mathbf{a}\,(\mathbf{b} \wedge \mathbf{c}).$$

In fact, expanding the determinant in the elements of the first row, we obtain

$$(\mathbf{abc}) = a_1 (b_2c_3 - b_3c_2) + a_2 (b_3c_1 - b_1c_3) + a_3 (b_1c_2 - b_2c_1) = \mathbf{a}\,(\mathbf{b} \wedge \mathbf{c}).$$

Interchanging two rows changes the sign. It follows that the *scalar triple product changes sign when two linear multipliers are interchanged, but a cyclic permutation of the multipliers does not change the sign*, i.e.,

$$(\mathbf{abc}) = -(\mathbf{bac}) = -(\mathbf{acb}) = -(\mathbf{cba}).$$

But

$$(\mathbf{abc}) = (\mathbf{bca}) = (\mathbf{cab}).$$

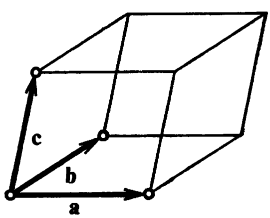

Fig. 59

It follows from the representation of the scalar triple product of vectors

$$(\mathbf{abc}) = \mathbf{a}\,(\mathbf{b} \wedge \mathbf{c})$$

that *it is zero if and only if at least one of the vectors is zero, or the vectors are coplanar.*

The scalar triple product of non-zero non-coplanar vectors has a simple geometrical meaning. Namely, *if the vectors have a common origin, then their scalar triple product is, up to a sign, equal to the volume of the parallelepiped constructed on these vectors* (Fig. 59). In fact,

$$|(\mathbf{abc})| = |(\mathbf{a} \wedge \mathbf{b})\,\mathbf{c}| = |S\,(\mathbf{ec})| = S\,|(\mathbf{ec})| = SH,$$

where S is the area of the base of the parallelepiped, H is its altitude, and **e** is the unit vector perpendicular to the base.

By way of exercise we will now find the volume of the tetrahedron with vertices at points $A_1(x_1, y_1, z_1)$, $A_2(x_2, y_2, z_2)$, $A_3(x_3, y_3, z_3)$, $A_4(x_4, y_4, z_4)$. The volume of the parallelepiped constructed on $\overrightarrow{A_1A_2}$, $\overrightarrow{A_1A_3}$, $\overrightarrow{A_1A_4}$ is six times the volume of the tetrahedron. Therefore, the volume of the tetrahedron is

$$V=\frac{1}{6}\left|(\overrightarrow{A_1A_2}\ \overrightarrow{A_1A_3}\ \overrightarrow{A_1A_4})\right|=\frac{1}{6}\left\|\begin{matrix} x_2-x_1 & x_3-x_1 & x_4-x_1 \\ y_2-y_1 & y_3-y_1 & y_4-y_1 \\ z_2-z_1 & z_3-z_1 & z_4-z_1 \end{matrix}\right\|. \quad (*)$$

This expression can be represented in a more symmetric form

$$V=\frac{1}{6}\left\|\begin{matrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{matrix}\right\|.$$

In fact, if we subtract the first column of this determinant from the others and expand it in the elements of the first row, we will arrive at (*).

7. Affine Cartesian Coordinates

The Cartesian coordinates we have used so far are called rectangular, because the coordinate axes form right angles with one another. But along with the rectangular coordinates in geometry and its applications some use is also made of the so-called affine (or oblique) coordinates. They can be introduced as follows.

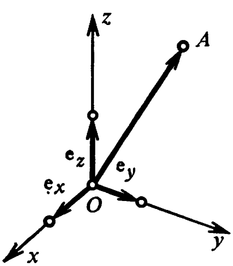

Fig. 60

Let us draw from an arbitrary point O in space three straight lines Ox, Oy, Oz not lying in one plane, and lay off on each of them from the point O three non-zero vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ (Fig. 60). According to Sec. 4, any vector $\overline{OA}$ allows a unique representation of the form

$$\overrightarrow{OA}=x\mathbf{e}_x+y\mathbf{e}_y+z\mathbf{e}_z.$$

The *numbers* x, y, z are called *affine Cartesian coordinates* of a point A.

The straight lines Ox, Oy, Oz are termed the axes of coordinates. Ox is the x-axis, Oy is the y-axis, and Oz is the z-axis. The planes Oxy, Oyz, Oxz are called the *coordinate planes*: Oxy is the xy-plane, Oyz is the yz-plane, and Oxz is the xz-plane.

Each of the coordinate axes is divided by the point O (i.e., by the *origin of coordinates*) into two semi-axes. The semi-axes whose directions coincide with the directions of the vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ are said to be *positive*, the others *negative*.

Geometrically the coordinates of the point A are obtained in the following way. We draw through the point A a plane parallel to the yz-plane. It intersects the x-axis at a point A_x (Fig. 61). Then the absolute value of the coordinate x of the point A is equal to the length of the line segment OA_x measured by the unit length $|\mathbf{e}_x|$. It is positive if A_x belongs to the positive semi-axis x, and is nega-

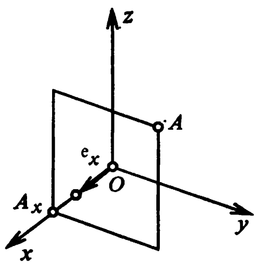

Fig. 61

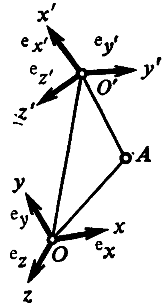

Fig. 62

tive if A_x belongs to the negative semi-axis x. The other two coordinates of the point (y and z) are determined by a similar construction.

If the coordinate axes are mutually perpendicular, and $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ are unit vectors, then the coordinates are *rectangular Cartesian*. In what follows, we, as a rule, shall use the rectangular Cartesian coordinates. Each case of application of affine Cartesian coordinates will be specified.

8. Transformation of Coordinates

Let two systems of affine coordinates xyz and $x'y'z'$ be introduced in space (Fig. 62). *Express the coordinates of an arbitrary point A with respect to $x'y'z'$ in terms of its coordinates with respect to xyz.*

We have

$$\overrightarrow{O'A} = x'\mathbf{e}_{x'} + y'\mathbf{e}_{y'} + z'\mathbf{e}_{z'},$$

$$\overrightarrow{O'O} = x'_0\mathbf{e}_{x'} + y'_0\mathbf{e}_{y'} + z'_0\mathbf{e}_{z'},$$

$$\overrightarrow{OA} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z,$$

$$\overrightarrow{O'A} = \overrightarrow{O'O} + \overrightarrow{OA} = (x'_0\mathbf{e}_{x'} + y'_0\mathbf{e}_{y'} + z'_0\mathbf{e}_{z'}) + (x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z).$$

The vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ allow a unique representation in terms of the vectors $\mathbf{e}_{x'}$, $\mathbf{e}_{y'}$, $\mathbf{e}_{z'}$:

$$\left.\begin{aligned} \mathbf{e}_x &= \alpha_{11}\mathbf{e}_{x'} + \alpha_{12}\mathbf{e}_{y'} + \alpha_{13}\mathbf{e}_{z'}, \\ \mathbf{e}_y &= \alpha_{21}\mathbf{e}_{x'} + \alpha_{22}\mathbf{e}_{y'} + \alpha_{23}\mathbf{e}_{z'}, \\ \mathbf{e}_z &= \alpha_{31}\mathbf{e}_{x'} + \alpha_{32}\mathbf{e}_{y'} + \alpha_{33}\mathbf{e}_{z'}, \end{aligned}\right\} \qquad (*)$$

where α_{ij} are the coordinates of the vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ relative to the basis $\mathbf{e}_{x'}$, $\mathbf{e}_{y'}$, $\mathbf{e}_{z'}$.

Substituting these expressions into the formula for $\overrightarrow{O'A}$, we get

$$\begin{aligned} \overrightarrow{O'A} = {} & (x_0' + \alpha_{11}x + \alpha_{21}y + \alpha_{31}z)\,\mathbf{e}_{x'} \\ & + (y_0' + \alpha_{12}x + \alpha_{22}y + \alpha_{32}z)\,\mathbf{e}_{y'} \\ & + (z_0' + \alpha_{13}x + \alpha_{23}y + \alpha_{33}z)\,\mathbf{e}_{z'}. \end{aligned}$$

The expressions in parentheses are the coordinates of the vector $\overrightarrow{O'A}$ relative to the basis $\mathbf{e}_{x'}$, $\mathbf{e}_{y'}$, $\mathbf{e}_{z'}$, i.e., the coordinates of the point A in the system $x'y'z'$. We get the required formulas:

$$\left.\begin{aligned} x' &= \alpha_{11}x + \alpha_{21}y + \alpha_{31}z + x_0', \\ y' &= \alpha_{12}x + \alpha_{22}y + \alpha_{32}z + y_0', \\ z' &= \alpha_{13}x + \alpha_{23}y + \alpha_{33}z + z_0'. \end{aligned}\right\} \qquad (**)$$

The coefficients of these formulas are α_{11}, α_{12}, α_{13}, the coordinates of the vector $\mathbf{e}_x$ relative to the basis $\mathbf{e}_{x'}$, $\mathbf{e}_{y'}$, $\mathbf{e}_{z'}$, α_{21}, α_{22}, α_{23} the coordinates of the vector $\mathbf{e}_y$, α_{31}, α_{32}, α_{33} the coordinates of the vector $\mathbf{e}_z$ and x_0', y_0', z_0' the coordinates of the point O in the coordinate system $x'y'z'$.

We note that the determinant

$$\Delta = \begin{vmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{vmatrix} \neq 0.$$

Indeed, one can directly check that

$$(\mathbf{e}_x\mathbf{e}_y\mathbf{e}_z) = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} (\mathbf{e}_{x'}\mathbf{e}_{y'}\mathbf{e}_{z'}).$$

Since $(\mathbf{e}_x\mathbf{e}_y\mathbf{e}_z) \neq 0$, then $\Delta \neq 0$.

For all systems of coordinates $x'y'z'$ which can be continuously transformed into one another the determinant has one and the same sign. (The continuity of changing a system of coordinates is understood as the continuity of changing the origin of coordinates O' and the basis $\mathbf{e}_{x'}$, $\mathbf{e}_{y'}$, $\mathbf{e}_{z'}$.) Indeed, since $(\mathbf{e}_x\mathbf{e}_y\mathbf{e}_z)$ is non-zero, Δ is also non-zero. Besides, since Δ changes continuously, it cannot attain values of different signs.

If $\Delta \neq 0$, then the system of formulas (**) may always be interpreted as a transformation from a coordinate system $x'y'z'$ to the coordinate system xyz whose origin is at point (x_0', y_0', z_0') and the basis vectors are expressed in terms of the basis vectors of the system $x'y'z'$ by the formulas (*).

If both systems of coordinates xyz and $x'y'z'$ are rectangular, then the coefficients of the formulas (**) satisfy the orthogonality conditions

$$\left.\begin{array}{ll} \alpha_{11}^2+\alpha_{12}^2+\alpha_{13}^2=1, & \alpha_{11}\alpha_{21}+\alpha_{12}\alpha_{22}+\alpha_{13}\alpha_{23}=0, \\ \alpha_{21}^2+\alpha_{22}^2+\alpha_{23}^2=1, & \alpha_{21}\alpha_{31}+\alpha_{22}\alpha_{32}+\alpha_{23}\alpha_{33}=0, \\ \alpha_{31}^2+\alpha_{32}^2+\alpha_{33}^2=1, & \alpha_{31}\alpha_{11}+\alpha_{32}\alpha_{12}+\alpha_{33}\alpha_{13}=0, \end{array}\right\} \qquad (***)$$

which are obtained when the formulas (*) and the following relationships

$$\begin{array}{ll} \mathbf{e}_x^2=\mathbf{e}_y^2=\mathbf{e}_z^2=1, & \mathbf{e}_x\mathbf{e}_y=\mathbf{e}_y\mathbf{e}_z=\mathbf{e}_z\mathbf{e}_x=0, \\ \mathbf{e}_{x'}^2=\mathbf{e}_{y'}^2=\mathbf{e}_{z'}^2=1, & \mathbf{e}_{x'}\mathbf{e}_{y'}=\mathbf{e}_{y'}\mathbf{e}_{z'}=\mathbf{e}_{z'}\mathbf{e}_{x'}=0, \end{array}$$

are used.

Conversely, if the conditions (***) are fulfilled, then the formulas (**) can always be interpreted as a transformation from a rectangular $x'y'z'$-coordinate system to the system of rectangular coordinates xyz whose origin is at the point (x_0', y_0', z_0') and the basis vectors are specified by the formulas (*). By virtue of the conditions (***) the basis vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ are unit vectors and are pairwise perpendicular.

In the case of rectangular Cartesian coordinates xyz and $x'y'z$, we have $\Delta = \pm 1$, where $\Delta = +1$ if one system of coordinates can be translated into the other system. If this cannot be done *without reflection*, then $\Delta = -1$.

9. Equations of a Surface and a Curve in Space

Suppose we have a surface (Fig. 63).

The equation

$$f(x, y, z) = 0 \qquad (*)$$

is called the *equation of a surface in implicit form* if the coordinates of any point of the surface satisfy this equation. And conversely, any three numbers x, y, z, which satisfy the equation (*), represent the coordinates of one of the points of the surface.

Simultaneous equations

$$x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v), \qquad (**)$$

which specify the coordinates of points of the surface as functions of two parameters (u, v) are called the *parametric equations of a surface.*

Eliminating the parameters u, v from the equations (**), we can obtain the equation of a surface in implicit form.

Write the equation for an arbitrary sphere in rectangular Cartesian coordinates xyz.

Let (x_0, y_0, z_0) be the centre of the sphere, and R its radius. Each point (x, y, z) of the sphere is at a distance R from the centre, and, consequently, satisfies the equation

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 - R^2 = 0. \qquad (***)$$

Conversely, any point (x, y, z) which satisfies the equation (***) is

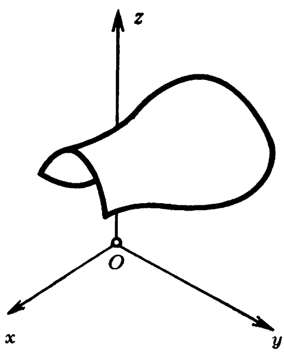

Fig. 63

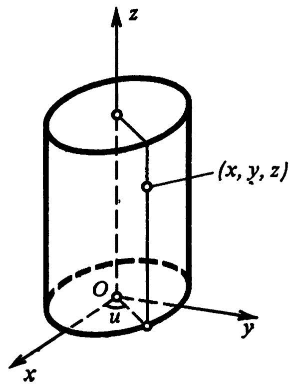

Fig. 64

at a distance R from (x_0, y_0, z_0) and, consequently, belongs to the sphere. The equation (***) is, by definition, the equation of a sphere.

Write the equation for a circular cylinder with the axis Oz and radius R (Fig. 64).

Let us take as the parameters u, v, characterizing the position of the point (x, y, z) on the cylinder, the coordinate z (v) and the angle (u) made by the plane passing through the z-axis and the point (x, y, z) with the xz-plane. We then get

$$x = R \cos u, \quad y = R \sin u, \quad z = v,$$

which are the *parametric equations of the cylinder.*

Squaring the first two equations and adding them termwise, we get the *equation of the cylinder in implicit form:*

$$x^2 + y^2 = R^2.$$

Suppose we have a curve in space. The simultaneous equations

$$f_1(x, y, z) = 0, \quad f_2(x, y, z) = 0$$

are called the *equations of a curve in implicit form* if the coordinates of each point of the curve satisfy both equations. And conversely, any three numbers which satisfy both equations represent the coordinates of some point on the curve.

Simultaneous equations

$$x = \varphi_1(t), \quad y = \varphi_2(t), \quad z = \varphi_3(t),$$

which specify the coordinates of points of the curve as functions of some parameter (t) are called the *equations of a curve in parametric form*.

Two surfaces intersect, as a rule, along a curve. Obviously, if the surfaces are specified by the equations $f_1(x, y, z) = 0$ and $f_2(x, y, z) = 0$, then the curve along which they intersect is given by simultaneous equations

$$f_1(x, y, z) = 0, \quad f_2(x, y, z) = 0.$$

Let us write the equation for an arbitrary circle in space. Any circle can be represented as an intersection of two spheres. Consequently, any circle can be specified by a system of equations

$$\left.\begin{aligned} (x-a_1)^2+(y-b_1)^2+(z-c_1)^2-R_1^2=0, \\ (x-a_2)^2+(y-b_2)^2+(z-c_2)^2-R_2^2=0. \end{aligned}\right\}$$

As a rule, a curve and a surface intersect at separate points. If the surface is specified by the equation $f(x, y, z) = 0$, and the curve by the equations $f_1(x, y, z) = 0$ and $f_2(x, y, z) = 0$, then the points of intersection of the curve and the surface satisfy the following simultaneous equations:

$$f(x, y, z) = 0, \quad f_1(x, y, z) = 0, \quad f_2(x, y, z) = 0.$$

Solving these equations we find the coordinates of the point of intersection.

EXERCISES TO CHAPTER V

1. Given points A (1, 2, 3), B (0, 1, 2), C (0, 0, 3), D (1, 2, 0). Which of these points lie (a) in the xy-plane, (b) on the z-axis, (c) in the yz-plane?

2. Given the point A (1, 2, 3), find the foot of the perpendiculars dropped from this point on the coordinate axes and coordinate planes.

3. Find the distances from a point (1, 2, —3) to (a) coordinate planes, (b) coordinate axes, (c) origin of coordinates.

4. In the xy-plane find a point D (x, y, 0) equidistant from three given points A (0, 1, —1), B (—1, 0, 1), C (0, —1, 0).

5. Find points equidistant from points (0, 0, 1), (0, 1, 0), (1, 0, 0) and separated from the yz-plane by a distance of 2.

6. On the x-axis find a point C (x, 0, 0) equidistant from two points A (1, 2, 3), B (—2, 1, 3).

7. Form the equation of the locus of points equidistant from the point A (1, 2, 3) and the origin of coordinates.

8. Prove that a quadrilateral $ABCD$ with vertices at A (1, 3, 2), B (0, 2, 4), C (1, 1, 4), D (2, 2, 2) is a parallelogram.

9. Given four points: A (6, 7, 8), B (8, 2, 6), C (4, 3, 2), D (2, 8, 4), show that they are vertices of a rhombus.

10. Given one end of a line segment A (2, 3, —1) and its midpoint C (1, 1, 1), find the other end B (x, y, z) of the segment.

11. Given the coordinates of three vertices of a parallelogram $ABCD$: A (2, 3, 2), B (0, 2, 4), C (4, 1, 0), find the coordinates of the fourth vertice D and the point E of intersection of the diagonals.

12. Given points (1, 2, 3), (0, —1, 2), (1, 0, —3), find the points symmetric to the given ones about the coordinate planes.

13. Given points (1, 2, 3), (0, —1, 2), (1, 0, —3), find the points symmetric to the given ones about the origin of coordinates.

14. Find the values of a, b, c in the formulas of translation $x' = x + a$, $y' = y + b$, $z' = z + c$, if under this translation the point A (1, 0, 2) changes into A' (2, 1, 0).

15. Under a translation the point A (2, 1, —1) changes into A' (1, —1, 0). Find the point to which the origin is moved.

16. Given points A (2, 7, —3), B (1, 0, 3), C (—3, —4, 5), D (—2, 3, —1). Find equal vectors among $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{DC}$, $\overrightarrow{AD}$, $\overrightarrow{AC}$, and $\overrightarrow{BD}$.

17. Given points A (1, 0, 1), B (—1, 1, 2), C (0, 2, —1). Find the point D (x, y, z), if $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal.

18. Find the point D in exercise 17, if the sum of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is zero.

19. Given the vectors $\overrightarrow{(2, n, 3)}$ and $\overrightarrow{(3, 2, m)}$, find at which m and n these vectors will be collinear.

20. Given $\mathbf{a}$ (1, 2, 3), find the vector collinear with $\mathbf{a}$ such that its origin is at A (1, 1, 1) and the terminus B in the xy-plane.

21. Given $\mathbf{a}$ (2, —1, 3) and $\mathbf{b}$ (1, 3, n), find at what n these vectors will be perpendicular.

22. Given points A (1, 0, 1), B (—1, 1, 2), C (0, 2, —1), find on the z-axis a point D (0, 0, c) such that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are perpendicular.

23. The vectors $\mathbf{a}$ and $\mathbf{b}$ form an angle of 60°, and the vector $\mathbf{c}$ is perpendicular to them. Find the magnitude of $\mathbf{a} + \mathbf{b} + \mathbf{c}$.

24. The vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ of unit length form with one another an angle of 60°. Find the angle φ between the vectors (a) $\mathbf{a}$ and $\mathbf{b} + \mathbf{c}$, (b) $\mathbf{a}$ and $\mathbf{b} - \mathbf{c}$.

25. Given points A (0, 1, —1), B (1, —1, 2), C (3, 1, 0), D (2, —3, 1), find the cosine of the angle φ between $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

26. Given points A (0, 1, —1), B (1, —1, 2), C (3, 1, 0). Find the cosine of the angle C of the triangle ABC.

27. Show that if the vectors **a** and **b** are perpendicular to the vector **c**, then

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = 0.$$

28. Show that if the vector **b** is perpendicular to **c**, and the vector **a** is parallel to the vector **c**, then

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = \mathbf{b}\,(\mathbf{ac}).$$

29. Show that for an arbitrary vector **a** and a vector **b** which is perpendicular to **c**

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = \mathbf{b}\,(\mathbf{ac}).$$

30. Show that for any three vectors **a**, **b** and **c**

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = \mathbf{b}\,(\mathbf{ac}) - \mathbf{a}\,(\mathbf{bc}).$$

31. Find the area of the base of a triangular pyramid whose lateral edges are equal to l, and the vertex angles are equal to α, β, γ.

32. Making note that

$$((\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c})\,\mathbf{d} = (\mathbf{a} \wedge \mathbf{b})\,(\mathbf{c} \wedge \mathbf{d}),$$

derive the identity

$$(\mathbf{a} \wedge \mathbf{b})\,(\mathbf{c} \wedge \mathbf{d}) = \begin{vmatrix} \mathbf{ac} & \mathbf{ad} \\ \mathbf{bc} & \mathbf{bd} \end{vmatrix}.$$

33. With the aid of the identity

$$(\mathbf{a} \wedge \mathbf{b})\,(\mathbf{c} \wedge \mathbf{b}) = (\mathbf{ac})\,\mathbf{b}^2 - (\mathbf{ab})\,(\mathbf{bc})$$

derive the formula of spherical trigonometry

$$\sin\alpha \sin\gamma \cos B = \cos\beta - \cos\gamma \cos\alpha,$$

where α, β, γ are the sides of a triangle on a unit sphere, and B is the angle of this triangle opposite the side β.

34. Derive the identity

$$(\mathbf{a} \wedge \mathbf{b}) \wedge (\mathbf{c} \wedge \mathbf{d}) = \mathbf{b}\,(\mathbf{acd}) - \mathbf{a}\,(\mathbf{bcd}).$$

35. Show that for any four vectors **a**, **b**, **c**, **d**

$$\mathbf{b}\,(\mathbf{acd}) - \mathbf{a}\,(\mathbf{bcd}) + \mathbf{d}\,(\mathbf{cab}) - \mathbf{c}\,(\mathbf{dab}) = 0.$$

36. Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be any three vectors satisfying the condition

$$(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) \neq 0.$$

Show that any vector **r** allows the representation

$$\mathbf{r} = \frac{(\mathbf{r}\mathbf{e}_2\mathbf{e}_3)\,\mathbf{e}_1}{(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)} + \frac{(\mathbf{r}\mathbf{e}_3\mathbf{e}_1)\,\mathbf{e}_2}{(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)} + \frac{(\mathbf{r}\mathbf{e}_1\mathbf{e}_2)\,\mathbf{e}_3}{(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)}.$$

37. Show that the solution of the following system of vector equations

$$(\mathbf{rab}) = \gamma, \quad (\mathbf{rbc}) = \alpha, \quad (\mathbf{rca}) = \beta,$$

where **a**, **b**, **c** are the given vectors satisfying the condition

$$(\mathbf{abc}) \neq 0,$$

and **r** is the required vector, can be written in the form

$$\mathbf{r} = \frac{1}{(\mathbf{abc})}(\mathbf{a}\alpha + \mathbf{b}\beta + \mathbf{c}\gamma).$$

38. Show that if $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and **r** are any four vectors satisfying a single condition $(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) \neq 0$, then the following identity takes place

$$\mathbf{r} = \frac{(\mathbf{e}_1 \wedge \mathbf{e}_2)(\mathbf{r}\mathbf{e}_3)}{(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)} + \frac{(\mathbf{e}_2 \wedge \mathbf{e}_3)(\mathbf{r}\mathbf{e}_1)}{(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)} + \frac{(\mathbf{e}_3 \wedge \mathbf{e}_1)(\mathbf{r}\mathbf{e}_2)}{(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)}.$$

39. Show that the solution of the system of simultaneous vector equations

$$\mathbf{ax} = \alpha, \quad \mathbf{bx} = \beta, \quad \mathbf{cx} = \gamma,$$

where **a**, **b**, **c** are the given vectors and **x** is the required vector satisfying the condition $(\mathbf{abc}) \neq 0$, can be written in the form

$$\mathbf{x} = \frac{(\mathbf{a} \wedge \mathbf{b})\gamma + (\mathbf{b} \wedge \mathbf{c})\alpha + (\mathbf{c} \wedge \mathbf{a})\beta}{(\mathbf{abc})}.$$

40. Show that $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ are coplanar if and only if

$$\begin{vmatrix} \mathbf{r}_1\mathbf{r}_1 & \mathbf{r}_1\mathbf{r}_2 & \mathbf{r}_1\mathbf{r}_3 \\ \mathbf{r}_2\mathbf{r}_1 & \mathbf{r}_2\mathbf{r}_2 & \mathbf{r}_2\mathbf{r}_3 \\ \mathbf{r}_3\mathbf{r}_1 & \mathbf{r}_3\mathbf{r}_2 & \mathbf{r}_3\mathbf{r}_3 \end{vmatrix} = 0.$$

41. Show that for any four vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ and $\mathbf{r}_4$

$$\begin{vmatrix} \mathbf{r}_1\mathbf{r}_1 & \mathbf{r}_1\mathbf{r}_2 & \mathbf{r}_1\mathbf{r}_3 & \mathbf{r}_1\mathbf{r}_4 \\ \mathbf{r}_2\mathbf{r}_1 & \mathbf{r}_2\mathbf{r}_2 & \mathbf{r}_2\mathbf{r}_3 & \mathbf{r}_2\mathbf{r}_4 \\ \mathbf{r}_3\mathbf{r}_1 & \mathbf{r}_3\mathbf{r}_2 & \mathbf{r}_3\mathbf{r}_3 & \mathbf{r}_3\mathbf{r}_4 \\ \mathbf{r}_4\mathbf{r}_1 & \mathbf{r}_4\mathbf{r}_2 & \mathbf{r}_4\mathbf{r}_3 & \mathbf{r}_4\mathbf{r}_4 \end{vmatrix} = 0.$$

42. Let l_1, l_2, l_3 and l_4 be four rays drawn from one point and α_{ij} be the angle between rays l_i and l_j. Prove the identity

$$\begin{vmatrix} 1 & \cos\alpha_{12} & \cos\alpha_{13} & \cos\alpha_{14} \\ \cos\alpha_{21} & 1 & \cos\alpha_{23} & \cos\alpha_{24} \\ \cos\alpha_{31} & \cos\alpha_{32} & 1 & \cos\alpha_{34} \\ \cos\alpha_{41} & \cos\alpha_{42} & \cos\alpha_{43} & 1 \end{vmatrix} = 0.$$

43. Show that the coordinates of the vector **r** relative to the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are given by the equalities

$$\lambda_1 = \frac{(\mathbf{r}\mathbf{e}_2\mathbf{e}_3)}{(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)}, \quad \lambda_2 = \frac{(\mathbf{r}\mathbf{e}_3\mathbf{e}_1)}{(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)}, \quad \lambda_3 = \frac{(\mathbf{r}\mathbf{e}_1\mathbf{e}_2)}{(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)}.$$

44. Show that the coordinates of the vector **r** relative to the basis $(\mathbf{e}_2 \wedge \mathbf{e}_3)$, $(\mathbf{e}_3 \wedge \mathbf{e}_1)$, $(\mathbf{e}_1 \wedge \mathbf{e}_2)$ are, respectively,

$$\lambda_1 = \frac{\mathbf{r}\mathbf{e}_1}{(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)}, \quad \lambda_2 = \frac{\mathbf{r}\mathbf{e}_2}{(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)}, \quad \lambda_3 = \frac{\mathbf{r}\mathbf{e}_3}{(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)}.$$

45. Decomposing the vectors **a**, **b**, **c** into the orthogonal basis, prove the identity

$$(\mathbf{abc})^2 = \begin{vmatrix} \mathbf{aa} & \mathbf{ab} & \mathbf{ac} \\ \mathbf{ba} & \mathbf{bb} & \mathbf{bc} \\ \mathbf{ca} & \mathbf{cb} & \mathbf{cc} \end{vmatrix},$$

using the determinant multiplication theorem.

46. Prove the identity

$$(\mathbf{a} \wedge \mathbf{b},\ \mathbf{b} \wedge \mathbf{c},\ \mathbf{c} \wedge \mathbf{a}) = (\mathbf{abc})^2.$$

47. Show that the volume of a triangular pyramid with the lateral edges a, b, c and face angles α, β, γ is

$$V = \frac{1}{6} abc \begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix}^{1/2}.$$

48. Find the distance between two points in affine coordinates if the positive axes form pairwise the angles α, β, γ, and $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ are unit vectors.

49. Find the centre of a sphere circumscribed about a tetrahedron with the vertices $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, $(0, 0, 0)$.

50. Prove that the straight lines joining the midpoints of the opposite edges of a tetrahedron intersect at one point. Express the coordinates of this point in terms of the coordinates of vertices of the tetrahedron.

51. Prove that the straight lines joining the vertices of a tetrahedron to the centroids of the opposite faces intersect at one point. Express its coordinates in terms of the coordinates of the vertices of the tetrahedron.

52. Let A_i (x_i, y_i, z_i) be the vertices of a tetrahedron. Show that the points with the coordinates

$$x = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4,$$
$$y = \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 + \lambda_4 y_4,$$
$$z = \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 + \lambda_4 z_4$$

are located inside the tetrahedron if $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$, $\lambda_4 > 0$, $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$.

53. For four points $A_i\,(x_i, y_i, z_i)$ to lie in one plane it is necessary and sufficient that

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

Prove this.

54. Show that the surface specified by the equation

$$x^2 + y^2 + z^2 + 2ax + 2by + 2cz + d = 0,$$

is a sphere if $a^2 + b^2 + c^2 - d > 0$. Find the coordinates of its centre and radius.

55. A circle is specified by the intersection of two spheres

$$\left.\begin{aligned} f_1(x, y, z) &= x^2 + y^2 + z^2 + 2a_1x + 2b_1y + 2c_1z + d_1 = 0, \\ f_2(x, y, z) &= x^2 + y^2 + z^2 + 2a_2x + 2b_2y + 2c_2z + d_2 = 0. \end{aligned}\right\}$$

Show that any sphere passing through this circle can be specified by the equation

$$\lambda_1 f_1(x, y, z) + \lambda_2 f_2(x, y, z) = 0.$$

56. Show that the surface specified by an equation of the form $\varphi(x, y) = 0$ is cylindrical. It is generated by straight lines parallel to the z-axis.

57. Form the equation for a right circular cone with the axis Oz, vertex O, and the vertex angle equal to 2α.

58. Form the equation of a surface described by the midpoint of a line segment whose endpoints belong to the curves γ_1 and γ_2

$$\gamma_1\colon \begin{array}{l} z = ax^2, \\ y = 0, \end{array} \qquad \gamma_2\colon \begin{array}{l} z = by^2, \\ x = 0. \end{array}$$

59. Form the equation for a surface generated by a straight line which, intersecting the curves γ_1 and γ_2, is parallel to the yz-plane:

$$\gamma_1\colon \begin{array}{l} z = f(x), \\ y = a, \end{array} \qquad \gamma_2\colon \begin{array}{l} z = \varphi(x), \\ y = b, \end{array} \qquad (a \neq b).$$

60. Show that the curve

$$z = \varphi(x), \quad y = 0 \quad (x > 0),$$

when revolving about the z-axis, generates a surface specified by the equation

$$z = \varphi\left(\sqrt{x^2 + y^2}\right).$$

61. Show that a cylindrical surface, with the generators parallel to the z-axis, which passes through the curve

$$z = f(x), \quad z = \varphi(y),$$

is specified by the equation

$$f(x) - \varphi(y) = 0.$$

62. What forms do the formulas for changing the coordinates have if the xy-plane coincides with the $x'y'$-plane?

63. We know that in a certain coordinate system the equation

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{23}yz + 2a_{31}zx = R^2$$

specifies a sphere. Find the angles between the coordinate axes.

64. Suppose we have two xyz- and $x'y'z'$-coordinate systems with a common origin O. Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be the basis of the first system, and $\mathbf{e}_1 \wedge \mathbf{e}_2$, $\mathbf{e}_2 \wedge \mathbf{e}_3$, $\mathbf{e}_3 \wedge \mathbf{e}_1$ the basis of the second system. Derive formulas for changing from one system to the other.

65. The transformation from the xy-coordinate system to the $x'y'z'$-coordinate system having the same origin can be accomplished in three stages:

$$\text{I} \quad \left.\begin{aligned} x_1 &= x\cos\varphi - y\sin\varphi, \\ y_1 &= x\sin\varphi + y\cos\varphi, \\ z_1 &= z; \end{aligned}\right\}$$

$$\text{II} \quad \left.\begin{aligned} x_2 &= x_1, \\ y_2 &= y_1\cos\theta - z_1\sin\theta, \\ z_2 &= y_1\sin\theta + z_1\cos\theta; \end{aligned}\right\}$$

$$\text{III} \quad \left.\begin{aligned} x' &= x_2\cos\psi - y_2\sin\psi, \\ y' &= x_2\sin\psi + y_2\cos\psi, \\ z' &= z_2. \end{aligned}\right\}$$

The angles φ, θ, ψ are called *Euler's angles*. Find out their geometrical meaning.

Chapter VI

PLANE AND A STRAIGHT LINE IN SPACE

1. Equation of a Plane

Prove that *any plane in space is described by an equation of the form*

$$ax + by + cz + d = 0, \tag{*}$$

where a, b, c, d are constants. Conversely, any equation of the form (*) *is the equation of a certain plane.*

Proof. Let $A_0\,(x_0, y_0, z_0)$ be some point in the plane and $\mathbf{n}\,(a, b, c)$ a vector perpendicular to the plane (Fig. 65). Let then $A\,(x, y, z)$ be an arbitrary point in the plane. Then $\overrightarrow{A_0A}$ and n will be perpendicular, and hence their scalar product will be zero. Thus, any point in the plane obeys the equation

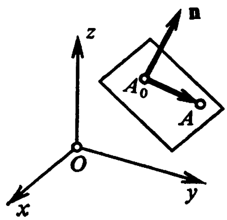

Fig. 65

$$a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0) = 0. \quad (**)$$

Conversely, if the point $A\,(x, y, z)$ satisfies this equation, then $\overrightarrow{A_0A} \cdot \mathbf{n} = 0$, and hence A lies in the plane. The equation (**) is thus the equation of our plane. It can be rewritten as

$$ax + by + cz + (-ax_0 - by_0 - cz_0) = 0.$$

We see that it has the form (*), which was to be proved. Suppose we have an equation

$$ax + by + cz + d = 0.$$

Show that it is now the equation of a certain plane. Let x_0, y_0, z_0 be some solution of this equation

$$ax_0 + by_0 + cz_0 + d = 0.$$

Using this relationship we can rewrite our equation as

$$ax + by + cz - ax_0 - by_0 - cz_0 = 0,$$

or

$$a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0) = 0.$$

And in this form, as we know, it is the equation of the plane passing through (x_0, y_0, z_0) and perpendicular to $\mathbf{n}\,(a, b, c)$. This proves the second statement.

Note that in *the equation of the plane*

$$ax + by + cz + d = 0$$

the coefficients a, b, c are the coordinates of the vector perpendicular to the plane.

It is well-known that the formulas of transformation from one Cartesian coordinate system to another are linear. Therefore, the equation of the plane in any, not necessarily rectangular, coordinate system is linear, i.e., has the form (*).

2. Position of a Plane Relative to a Coordinate System

Let us consider features specifying the position of a plane in space, relative to a coordinate system, if its equation has the following particular form:

1. $a = 0$, $b = 0$. Vector **n** (perpendicular to the plane) is parallel to the z-axis. The plane is parallel to the xy-plane and, in particular, coincides with it at the xy-plane if $d = 0$.

2. $b = 0$, $c = 0$. The plane is parallel to the yz-plane and coincides with it if $d = 0$.

3. $c = 0$, $a = 0$. The plane is parallel to the xz-plane and coincides with it if $d = 0$.

4. $a = 0$, $b \neq 0$, $c \neq 0$. A vector **n** is perpendicular to the x-axis: $\mathbf{e}_x\mathbf{n} = 0$. The plane is parallel to the x-axis, in particular, it passes through it if $d = 0$.

5. $a \neq 0$, $b = 0$, $c \neq 0$. The plane is parallel to the y-axis and passes through it if $d = 0$.

6. $a \neq 0$, $b \neq 0$, $c = 0$. The plane is parallel to the z-axis and passes through it if $d = 0$.

7. $d = 0$. The plane passes through the origin (whose coordinates 0, 0, 0 satisfy the equation of the plane).

If in the equation of the plane the coefficient of z is nonzero, then the equation can be solved for z. It becomes

$$z = px + qy + r.$$

The coefficients p and q in this equation are called *angular coefficients*.

3. Normal Form of Equations of the Plane

If a point A (x, y, z) belongs to the plane

$$ax + by + cz + d = 0, \tag{*}$$

then its coordinates satisfy the equation (*).

Let us consider what geometrical meaning has the expression

$$ax + by + cz + d$$

if the point A does not belong to the plane.

We drop a perpendicular from the point A onto the plane. Let A_0 (x_0, y_0, z_0) be the foot of the perpendicular. Since the point A_0 lies on the plane, then

$$ax_0 + by_0 + cz_0 + d = 0.$$

Whence

$$\begin{aligned} ax + by + cz + d &= a(x - x_0) + b(y - y_0) + c(z - z_0) \\ &= \mathbf{n}\cdot\overrightarrow{A_0A} = \pm|\mathbf{n}|\delta, \end{aligned}$$

where **n** is a vector perpendicular to the plane, with the coordinates a, b, c, and δ is the distance from the point A to the plane.

Thus

$$ax + by + cz + d$$

is positive on one side of the plane, and negative on the other, its absolute value being proportional to the distance from the point A to the plane. The proportionality factor

$$\pm |\mathbf{n}| = \pm \sqrt{a^2+b^2+c^2}.$$

If in the equation of the plane

$$a^2 + b^2 + c^2 = 1,$$

then

$$ax + by + cz + d$$

will be equal, up to a sign, to the distance from the point to the plane. In this case the plane is said to be specified by an *equation in the normal form.*

Obviously, to obtain the normal form of the equation of a plane (*), it is sufficient to divide it by

$$\pm \sqrt{a^2+b^2+c^2}.$$

4. Parallelism and Perpendicularity of Planes

Suppose we have two planes

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1 = 0, \\ a_2x + b_2y + c_2z + d_2 = 0. \end{aligned}\right\} \qquad (*)$$

Consider the condition under which these planes are: (a) *parallel,* (b) *mutually perpendicular.*

Since a_1, b_1, c_1 are the coordinates of the vector $\mathbf{n}_1$ perpendicular to the first plane, and a_2, b_2, c_2 are the coordinates of the vector $\mathbf{n}_2$ which is perpendicular to the second plane, the planes are parallel if the vectors $\mathbf{n}_1$, $\mathbf{n}_2$ are parallel, i.e. if their coordinates are proportional:

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

Moreover, this condition is sufficient for parallelism of the planes if they are not coincident.

For the planes (*) to be mutually perpendicular it is necessary and sufficient that the mentioned vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are mutually perpendicular, which for non-zero vectors is equivalent to the condition

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = 0 \text{ or } a_1a_2 + b_1b_2 + c_1c_2 = 0.$$

Let the equations (*) specify two arbitrary planes. Find the angle made by these planes.

The angle θ between the planes is either equal to the angle between the vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ perpendicular to the planes, or together with it makes the angle of 180°. Thus, in any case

$$|\mathbf{n}_1 \cdot \mathbf{n}_2| = |\mathbf{n}_1| \, |\mathbf{n}_2| \cos\theta.$$

Whence

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}.$$

5. Equations of a Straight Line

Any straight line can be specified as an intersection of two planes. Consequently, *any straight line can be specified by the equations*

$$\left.\begin{array}{l} a_1x + b_1y + c_1z + d_1 = 0, \\ a_2x + b_2y + c_2z + d_2 = 0, \end{array}\right\} \qquad (*)$$

the first of which represents one plane and the second the other. Conversely, *any compatible system of two such independent equations represents the equations of a straight line.*

Let A_0 (x_0, y_0, z_0) be a fixed point on a straight line, A (x, y, z) an arbitrary point of a straight line, and $\mathbf{e}$ (k, l, m) a non-zero vector parallel to the straight line (Fig. 66). Then the vectors $\overrightarrow{A_0A}$ and $\mathbf{e}$ are parallel and, consequently, their coordinates are proportional, i.e.

$$\frac{x-x_0}{k} = \frac{y-y_0}{l} = \frac{z-z_0}{m}. \qquad (**)$$

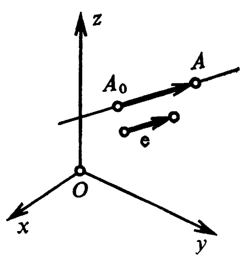

Fig. 66

Such an equation of a straight line is called *canonical* and is a particular case of (*), since it allows an equivalent

$$\frac{x-x_0}{k} = \frac{y-y_0}{l}, \quad \frac{y-y_0}{l} = \frac{z-z_0}{m},$$

corresponding to (*).

Suppose a straight line is represented by the equations (*). Let us form its equation in canonical form. For this purpose it is sufficient to find a point A_0 on the straight line and a vector $\mathbf{e}$ parallel to this line.

Any vector $\mathbf{e}$ (k, l, m) parallel to the straight line will be parallel to each of the planes (*), and vice versa. Consequently, k, l, m satisfy the equations

$$\left.\begin{array}{l} a_1k + b_1l + c_1m = 0, \\ a_2k + b_2l + c_2m = 0. \end{array}\right\} \qquad (***)$$

Thus, any solution of the system (*) can be taken as x_0, y_0, z_0 for the canonical equation of the straight line and any solution of (***) as the coefficients k, l, m, for instance

$$k=\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad l=\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \quad m=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

From the equation of a straight line in canonical form we can derive its equations in parametric form. Namely, assuming the common value of the three ratios of the canonical equation equal to t, we get

$$x = kt + x_0, \quad y = lt + y_0, \quad z = mt + z_0$$

which are the *parametric equations of a straight line.*

Let us find out what are the peculiarities of the position of a straight line relative to the coordinate system if some of the coefficients of the canonical equation are equal to zero.

Since the vector $\mathbf{e}$ (k, l, m) is parallel to the straight line, with $m = 0$ the line is parallel to the xy-plane ($\mathbf{e}\mathbf{e}_z = 0$), with $l = 0$ the line is parallel to the xz-plane, and with $k = 0$ it is parallel to the yz-plane.

If $k = 0$ and $l = 0$, then the straight line is parallel to the z-axis ($\mathbf{e} \parallel \mathbf{e}_z$); if $l = 0$ and $m = 0$, then it is parallel to the x-axis, and if $k = 0$ and $m = 0$, then the line is parallel to the y-axis.

6. Relative Position of a Straight Line and a Plane, of Two Straight Lines

Suppose we have a plane and a straight line respectively specified by the equations

$$ax + by + cz + d = 0,$$

$$\frac{x-x_0}{k} = \frac{y-y_0}{l} = \frac{z-z_0}{m}.$$

Since the vector $\overrightarrow{(a, b, c)}$ is perpendicular to the plane, and the vector $\overrightarrow{(k, l, m)}$ is parallel to the straight line, then the *straight line and the plane will be parallel if these vectors are perpendicular, i.e. if*

$$ak + bl + cm = 0. \qquad (*)$$

Moreover, if the point (x_0, y_0, z_0) belonging to the straight line satisfies the equation of the plane

$$ax_0 + by_0 + cz_0 + d = 0,$$

then the *straight line lies in the plane.*

The straight line and the plane are perpendicular if the vectors $\overrightarrow{(a, b, c)}$ *and* $\overrightarrow{(k, l, m)}$ *are parallel, i.e. if*

$$\frac{a}{k} = \frac{b}{l} = \frac{c}{m}. \qquad (**)$$

We can obtain the parallelism and perpendicularity conditions for a straight line and a plane if the straight line is represented by the intersection of the planes

$$a_1x + b_1y + c_1z + d_1 = 0,$$
$$a_2x + b_2y + c_2z + d_2 = 0.$$

It is sufficient to note that the vector with the components

$$k = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad l = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \quad m = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

is parallel to the straight line and make use of the conditions (*) and (**).

Suppose two straight lines are specified by the equations in canonical form

$$\left.\begin{aligned} \frac{x-x'}{k'} = \frac{y-y'}{l'} = \frac{z-z'}{m'}, \\ \frac{x-x''}{k''} = \frac{y-y''}{l''} = \frac{z-z''}{m''}. \end{aligned}\right\} \qquad (***)$$

Since the vector $\overrightarrow{(k', l', m')}$ is parallel to the first line, and the vector $\overrightarrow{(k'', l'', m'')}$ is parallel to the second line, then the *lines are parallel if*

$$\frac{k'}{k''} = \frac{l'}{l''} = \frac{m'}{m''}.$$

In particular, the straight lines coincide if a point of the first line, say (x', y', z'), satisfies the equation of the second line, i.e. if

$$\frac{x'-x''}{k''} = \frac{y'-y''}{l''} = \frac{z'-z''}{m''}.$$

The straight lines are perpendicular if the vectors $\overrightarrow{(k', l', m')}$ *and* $\overrightarrow{(k'', l'', m'')}$ *are perpendicular, i.e. if*

$$k'k'' + l'l'' + m'm'' = 0.$$

7. Basic Problems on Straight Lines and Planes

Form the equation of the straight line passing through two given points $A_1(x_1, y_1, z_1)$ and $A_2(x_2, y_2, z_2)$.

The vector $\mathbf{e}(x_2 - x_1, y_2 - y_1, z_2 - z_1)$ lies on the straight line. Accordingly, the straight line is given by the equations

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}.$$

Form the equation of the plane passing through three points $A_1(x_1, y_1, z_1)$, $A_2(x_2, y_2, z_2)$ and $A_3(x_3, y_3, z_3)$.

Let $A(x, y, z)$ be an arbitrary point in the plane. Then $\overrightarrow{A_1A_2}$, $\overrightarrow{A_1A_3}$, and $\overrightarrow{A_1A}$ are coplanar, and hence their scalar triple product is zero. From this we obtain the required equation

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0.$$

Form the equation of the plane passing through a point (x_0, y_0, z_0) and parallel to the plane

$$ax + by + cz + d = 0.$$

The desired equation will be

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

In fact, this plane passes through the given point and is parallel to the given plane.

Form the equation of the straight line passing through the point (x_0, y_0, z_0) parallel to the straight line

$$\frac{x-x'}{k} = \frac{y-y'}{l} = \frac{z-z'}{m}.$$

The desired equation is

$$\frac{x-x_0}{k} = \frac{y-y_0}{l} = \frac{z-z_0}{m}.$$

The straight line passing through (x_0, y_0, z_0) perpendicular to the plane

$$ax + by + cz + d = 0,$$

is given by

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}.$$

The plane perpendicular to the straight line

$$\frac{x-x'}{k}=\frac{y-y'}{l}=\frac{z-z'}{m},$$

and passing through the point (x_0, y_0, z_0) *is given by*

$$k(x-x_0)+l(y-y_0)+m(z-z_0)=0.$$

Form the equation of the plane passing through the point (x_0, y_0, z_0) *parallel to the straight lines*

$$\frac{x-x'}{k'}=\frac{y-y'}{l'}=\frac{z-z'}{m'},$$

$$\frac{x-x''}{k''}=\frac{y-y''}{l''}=\frac{z-z''}{m''}.$$

Since the vectors $\overrightarrow{(k', l', m')}$ and $\overrightarrow{(k'', l'', m'')}$ are parallel to the plane, then their vector product is perpendicular to the plane. Hence the desired equation

$$(x-x_0)\begin{vmatrix} l' & m' \\ l'' & m'' \end{vmatrix}+(y-y_0)\begin{vmatrix} m' & k' \\ m'' & k'' \end{vmatrix}+(z-z_0)\begin{vmatrix} k' & l' \\ k'' & l'' \end{vmatrix}=0,$$

or in shorthand

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ k' & l' & m' \\ k'' & l'' & m'' \end{vmatrix}=0.$$

EXERCISES TO CHAPTER VI

1. Find the line segments cut off by the plane $ax+by+cz+d=0$ on the coordinate axes, if $abcd\neq 0$.
2. Show that the line of intersection of the planes given by $a_1x+b_1y=d_1$, $a_2x+b_2y=d_2$ is parallel to the z-axis.
3. Show that the planes given by

$$ax+by+cz+d=0 \text{ and } ax+by+cz+d_1=0$$

have no points in common if $d\neq d_1$.
4. Show that any plane parallel to the plane $ax+by+cz+d=0$ is given by an equation of the form $ax+by+cz+d'=0$, where $d'\neq d$.
5. A plane is given by the equation $ax+by+cz+d=0$. What condition must the coordinates of $P(k, l, m)$ satisfy for the straight line passing through this point and the origin of coordinates to be perpendicular to the plane?

6. Given the point $P(k, l, m)$, find the equation of the plane passing through the origin of coordinates O and perpendicular to the straight line OP.

7. Find the point of intersection of three planes given by the equations

$$x + y + z = 1, \quad x - 2y = 0, \quad 2x + y + 3z + 1 = 0.$$

8. Show that the planes given by

$$x + y + z = 1, \ 2x + y + 3z + 1 = 0, \ x + 2z + 1 = 0$$

do not have a single point in common.

9. Under what condition is the plane given by $ax + by + cz + d = 0$ perpendicular to the xy-plane?

10. A plane is given by the equation $2x + 3y + z = 1$. Indicate a vector parallel to the plane.

11. A straight line is the line of intersection of the planes $2x + 3y + z = 1$, $x + y + z = 1$. Indicate a vector parallel to the straight line.

12. Form the equation of a plane given two points (x_1, y_1, z_1) and (x_2, y_2, z_2) situated symmetrically about it.

13. What is the locus of points whose coordinates satisfy the equation

$$(ax + by + cz + d)^2 - (\alpha x + \beta y + \gamma z + \delta)^2 = 0?$$

14. Show that the curve represented by the equations

$$\left. \begin{aligned} f(x, y, z) + a_1x + b_1y + c_1z + d_1 = 0, \\ f(x, y, z) + a_2x + b_2y + c_2z + d_2 = 0, \end{aligned} \right\}$$

is plane, i.e., all its points are in a certain plane.

15. Write an equation for the plane which passes through the circle of intersection of the two spheres

$$x^2 + y^2 + z^2 + ax + by + cz + d = 0,$$
$$x^2 + y^2 + z^2 + \alpha x + \beta y + \gamma z + \delta = 0.$$

16. Show that inversion transforms a sphere either into a sphere or into a plane.

17. Show that the equation of any plane which passes through the line of intersection of the planes

$$ax + by + cz + d = 0,$$
$$\alpha x + \beta y + \gamma z + \delta = 0,$$

can be represented in the form

$$\lambda(ax + by + cz + d) + \mu(\alpha x + \beta y + \gamma z + \delta) = 0.$$

18. Show that the plane passing through the three given points (x_i, y_i, z_i) $(i = 1, 2, 3)$ is specified by the equation

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

19. Find the conditions under which the plane

$$ax + by + cz + d = 0$$

intersects the positive x-axis (y, z).

20. Find the volume of the tetrahedron bounded by the coordinate planes and the plane

$$ax + by + cz + d = 0$$

if $abcd \neq 0$.

21. Prove that the points in space for which

$$|x| + |y| + |z| < a,$$

are inside an octahedron with centre at the origin and the vertices on the coordinate axes.

22. Given a plane σ by the equation in rectangular Cartesian coordinates

$$ax + by + cz + d = 0,$$

form an equation of the plane σ' symmetric to σ about the xy-plane (about the origin O).

23. Given a family of planes depending on the parameter λ

$$ax + by + cz + d + \lambda (\alpha x + \beta y + \gamma z + \delta) = 0,$$

find a plane parallel to the z-axis.

24. In the family of planes

$$(a_1x + b_1y + c_1z + d_1) + \lambda (a_2x + b_2y + c_2z + d_2) + \mu (a_3x + b_3y + c_3z + d_3) = 0$$

find the plane parallel to the xy-plane. The parameters of the family are λ and μ.

25. The planes specified by the equations in rectangular Cartesian coordinates

$$ax + by + cz + d = 0,$$
$$ax + by + cz + d' = 0,$$

where $d \neq d'$, have no points in common, hence, they are parallel. Find the distance between these planes.

26. The plane

$$ax + by + d = 0$$

is parallel to z-axis. Find the distance from the z-axis to this plane.

27. What is the locus of points whose distances from the two given planes are in a given ratio?

28. Form the equations of the planes parallel to the plane

$$ax + by + cz + d = 0$$

and located at a distance δ from it.

29. Show that the points in space satisfying the condition

$$|ax + by + cz + d| < \delta^2,$$

lie between the parallel planes

$$ax + by + cz + d \pm \delta^2 = 0.$$

30. Given are equations of the planes containing the faces of a tetrahedron and a point M specified by its coordinates. How do you find whether or not the point M lies inside the tetrahedron?

31. Derive formulas for the transition to a new system of rectangular Cartesian coordinates $x'y'z'$ if the new coordinate planes are specified in the old system by the equations

$$a_1x + b_1y + c_1z + d_1 = 0,$$
$$a_2x + b_2y + c_2z + d_2 = 0,$$
$$a_3x + b_3y + c_3z + d_3 = 0.$$

32. Find the angles formed by the plane

$$ax + by + cz + d = 0$$

and the coordinate axes.

33. Find the angle formed by the plane

$$z = px + qy + l$$

with the xy-plane.

34. Show that the area of a figure F in the plane

$$z = px + qy + l$$

and the area of its projection $\bar{F}$ onto the xy-plane are related as

$$S(F) = \sqrt{1 + p^2 + q^2}\, S(\bar{F}).$$

35. Under what condition does the plane

$$ax + by + cz + d = 0$$

intersect the x- and y-axes at equal angles? Under what condition does it intersect all three x, y- and z-axes?

36. Among the planes of a pencil

$$\lambda (a_1x + b_1y + c_1z + d_1) + \mu (a_2x + b_2y + c_2z + d_2) = 0$$

find the plane perpendicular to the plane

$$ax + by + cz + d = 0.$$

37. Let

$$a_1x + b_1y + c_1z + d_1 = 0,$$
$$a_2x + b_2y + c_2z + d_2 = 0,$$
$$a_3x + b_3y + c_3z + d_3 = 0$$

be the equations of three planes not parallel to a straight line. Show that any plane passing through the point of intersection of the given planes has the equation of the form:

$$\lambda_1 (a_1x + b_1y + c_1z + d_1) + \lambda_2 (a_2x + b_2y + c_2z + d_2) + \lambda_3 (a_3x + b_3y + c_3z + d_3) = 0.$$

38. Under what condition does a straight line represented by the equation in canonical form intersect the x-axis (y-axis, z-axis)? Under what condition is it parallel to the plane xy (yz, zx)?

39. Show that the locus of points equidistant from three pairwise non-parallel planes is a straight line.

40. Show that the locus of points equidistant from the vertices of a triangle is a straight line. Form its equations given the coordinates of the vertices of the triangle.

41. Show that two straight lines entirely lying on the surface pass through each point of the surface

$$z = axy.$$

42. If the straight lines specified by the equations

$$\left.\begin{aligned} a_1x+b_1y+c_1z+d_1&=0,\\ a_2x+b_2y+c_2z+d_2&=0 \end{aligned}\right\}$$

and

$$\left.\begin{aligned} a_3x+b_3y+c_3z+d_3&=0,\\ a_4x+b_4y+c_4z+d_4&=0, \end{aligned}\right\}$$

intersect, then

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0.$$

Show this.

43. Find the parallelism (perpendicularity) condition for the straight line

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1 = 0, \\ a_2x + b_2y + c_2z + d_2 = 0, \end{aligned}\right\}$$

and the plane

$$ax + by + cz + d = 0.$$

44. Find the parallelism (perpendicularity) condition for the straight lines

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1 = 0, \\ a_2x + b_2y + c_2z + d_2 = 0 \end{aligned}\right\}$$

and

$$\left.\begin{aligned} a_3x + b_3y + c_3z + d_3 = 0, \\ a_4x + b_4y + c_4z + d_4 = 0. \end{aligned}\right\}$$

45. Find the equation for a conical surface with the vertex (x_0, y_0, z_0), whose generators intersect the plane

$$ax + by + cz + d = 0$$

at an angle α.

46. Write the equation for a straight line passing through the point (x_0, y_0, z_0) and parallel to the planes

$$a_1x + b_1y + c_1z + d_1 = 0,$$
$$a_2x + b_2y + c_2z + d_2 = 0.$$

47. Form the equation of a conical surface with the vertex at point $(0, 0, 2R)$ if it passes through a circle specified by the intersection of the sphere

$$x^2 + y^2 + z^2 = 2Rz$$

with the plane

$$ax + by + cz + d = 0.$$

How does this conical surface intersect the xy-plane.

48. *Stereographic projection of a sphere* on a plane is defined as the projection from an arbitrary point of this sphere onto the tangent plane at a diametrically opposite point. Show that in stereographic projecting to the circles on the sphere there correspond circles and straight lines on the plane of projection.

49. Form the equation of a plane equidistant from two skew lines represented by the canonical equations.

50. Show that any plane passing through the straight line

$$a_1x + b_1y + c_1z + d_1 = 0,$$
$$a_2x + b_2y + c_2z + d_2 = 0,$$

is specified by an equation of the form

$$\lambda\,(a_1x + b_1y + c_1z + d_1) + \mu\,(a_2x + b_2y + c_2z + d_2) = 0.$$

51. Show that the plane passing through the straight line

$$\frac{x-x'}{k}=\frac{y-y'}{l}=\frac{z-z'}{m}$$

and the point (x_0, y_0, z_0), not lying on the line is specified by the equation

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ x'-x_0 & y'-y_0 & z'-z_0 \\ k & l & m \end{vmatrix}=0.$$

52. Show that any straight line intersecting the given lines:

$$\left.\begin{array}{l} a_1x+b_1y+c_1z+d_1=0, \\ a_2x+b_2y+c_2z+d_2=0, \end{array}\right\}$$
$$\left.\begin{array}{l} a_3x+b_3y+c_3z+d_3=0, \\ a_4x+b_4y+c_4z+d_4=0, \end{array}\right\}$$

is represented by the equations

$$\lambda\,(a_1x + b_1y + c_1z + d_1) + \mu\,(a_2x + b_2y + c_2z + d_2) = 0,$$
$$\lambda'\,(a_3x + b_3y + c_3z + d_3) + \mu'\,(a_4x + b_4y + c_4z + d_4) = 0.$$

53. Show that the conical surface generated by straight lines passing through the origin and intersecting the curve $\varphi(x, y) = 0$, $z = 1$ is specified by the equation

$$\varphi\left(\frac{x}{z}, \frac{y}{z}\right)=0.$$

Chapter VII

QUADRIC SURFACES

1. Special System of Coordinates

A quadric surface is defined as the locus of points in space whose Cartesian coordinates satisfy an equation of the form

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{23}yz + 2a_{13}xz + 2a_{14}x + 2a_{24}y + 2a_{34}z + a_{44} = 0. \quad (*)$$

This definition is obviously invariant of the system of coordinates chosen. Indeed, the equation of the surface in any other system of coordinates $x'y'z'$ is obtained from the equation (*) by replacing x, y and z by linear expressions with respect to x', y', z', and, consequently, in the coordinates x', y', z' will also have the form (*).

A plane intersects a quadric surface along a trace, which is described by a second-degree equation. Indeed, since the determination of surface is invariant with reference to the coordinate system chosen, we may regard the plane xy ($z = 0$) as a secant plane (a plane that intersects the surface). And this plane obviously intersects the surface along the second-order curve

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{14}x + 2a_{24}y + a_{44} = 0.$$

To study the geometrical properties of a quadric surface it is only natural to refer it to such a coordinate system in which its equation will have the simplest form.

Now *we are going to give a coordinate system in which the equation of the surface will be much simpler. Namely, the coefficients of yz, xz, and xy in the equation will be zero.*

Consider the function $F(A)$ of a point $A(x, y, z)$ defined in the entire space, except for the origin, by the equality

$$F(A) = \frac{a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{23}yz + 2a_{13}xz}{x^2 + y^2 + z^2}.$$

It is bounded on a unit sphere ($x^2 + y^2 + z^2 = 1$) and, consequently, reaches the absolute minimum at some point A_0. And since it is constant along any ray emanating from the origin ($F(\lambda x, \lambda y, \lambda z) = F(x, y, z)$), then at A_0 the function F reaches the absolute minimum of values with reference to the whole space (and not only on unit sphere).

Let us introduce new Cartesian coordinates x', y', z' with the origin O retained and assume the ray OA_0 to be the positive semi-axis z. As is known, the relation between the coordinates x, y, z and x', y', z' is established by the formulas of the form

$$\left.\begin{aligned} x &= \alpha_{11}x' + \alpha_{12}y' + \alpha_{13}z', \\ y &= \alpha_{21}x' + \alpha_{22}y' + \alpha_{23}z', \\ z &= \alpha_{31}x' + \alpha_{32}y' + \alpha_{33}z', \end{aligned}\right\} \qquad (**)$$

The equation of the surface in the new coordinates x', y', z' is obtained from the equation (*) upon replacing x, y, z by x', y', z' according to formulas (**) and has the form

$$a'_{11}x'^2 + a'_{22}y'^2 + a'_{33}z'^2 + 2a'_{12}x'y' + 2a'_{23}y'z' + 2a'_{13}x'z' + 2a'_{14}x' + 2a'_{24}y' + 2a'_{34}z' + a'_{44} = 0.$$

The function F in the new coordinates has the form

$$F(A)=\frac{a'_{11}x'^2+a'_{22}y'^2+a'_{33}z'^2+2a'_{12}x'y'+2a'_{23}y'z'+2a'_{13}x'z'}{x'^2+y'^2+z'^2}$$

and is obtained by replacing x, y, z in the old expression for F by x', y', z' also according to the formulas (**). The form of the denominator remains unchanged, since it represents the square of the distance of the point A from the origin which is expressed in both systems in the same way.

According to the chosen system of coordinates $x'y'z'$ the minimum of the function F is reached at $x'=0$, $y'=0$, $z'=1$. Therefore, if in the expression for F we put $x'=0$, $z'=1$, then we get a function of a single variable

$$f(y')=\frac{a'_{22}y'^2+2a'_{23}y'+a'_{33}}{1+y'^2},$$

which reaches the minimum at $y'=0$. Consequently,

$$\frac{df(y')}{dy'}=0 \text{ for } y'=0.$$

But

$$\left.\frac{df(y')}{dy'}\right|_{y'=0}=2a'_{23}.$$

Thus, the coefficient of $y'z'$ in the equation of the surface is equal to zero. It is shown in a similar way that the coefficient of $x'z'$ is also equal to zero.

Hence, the equation of the surface in the coordinate system $x'y'z'$ will be

$$a'_{11}x'^2+2a'_{12}x'y'+a'_{22}y'^2+2a'_{14}x'+2a'_{24}y'+2a'_{34}z' + a'_{33}z'^2+a'_{44}=0.$$

If now we introduce new coordinates x'', y'', z'' according to the formulas

$$x'=x''\cos\theta+y''\sin\theta,$$
$$y'=-x''\sin\theta+y''\cos\theta,$$
$$z'=z'',$$

then by appropriate choice of the angle θ we can obtain the coefficient of $x''y''$ also equal to zero.

And so, *there exists such a system of rectangular Cartesian coordinates in which the equation of the surface has the form*

$$a_{11}x^2+a_{22}y^2+a_{33}z^2+2a_1x+2a_2y+2a_3z+a=0.$$

2. Classification of Quadric Surfaces

As is shown in the preceding section, by changing to an appropriate system of coordinates the equation of a quadric surface can be reduced to the form

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_1x + 2a_2y + 2a_3z + a = 0. \qquad (*)$$

We shall distinguish three basic cases:

A: all coefficients of the squares of the coordinates in equation (*) are non-zero;

B: two coefficients are non-zero, and the third one, for instance a_{33}, is equal to zero;

C: one coefficient, say a_{33}, is non-zero, and two others are equal to zero.

In Case A, by changing to a new coordinate system according to the formulas

$$x' = x + \frac{a_1}{a_{11}}, \quad y' = y + \frac{a_2}{a_{22}}, \quad z' = z + \frac{a_3}{a_{33}},$$

which corresponds to the translation of the origin, we reduce the equation to the form

$$\alpha x'^2 + \beta y'^2 + \gamma z'^2 + \delta = 0.$$

Now we distinguish the following subcases of the case A:

A_1: $\delta = 0$. The surface is a *cone* either *imaginary* if α, β, γ are of the same sign, or *real* if among the numbers α, β, γ there are numbers having different signs.

A_2: $\delta \neq 0$, α, β, γ are of the same sign. The surface represents an *ellipsoid* either *imaginary* if α, β, γ, δ are of the same sign, or *real* if the sign of δ is opposite to that of α, β, γ.

A_3: $\delta \neq 0$, of the four coefficients α, β, γ, δ two coefficients are of one sign, the remaining two having the opposite sign. The surface is a *hyperboloid of one sheet.*

A_4: $\delta \neq 0$, one of the first three coefficients has a sign opposite to that of the remaining coefficients. The surface is a *two-sheeted hyperboloid.*

In Case B by transition to new coordinates according to the formulas

$$x' = x + \frac{a_1}{a_{11}}, \quad y' = y + \frac{a_2}{a_{22}}, \quad z' = z$$

we reduce the equation of the surface to the form

$$\alpha x'^2 + \beta y'^2 + 2pz' + q = 0.$$

Here we shall distinguish the following subcases:

B_1: $p = 0$, $q = 0$. The surface decomposes into a pair of planes

$$x' \pm \sqrt{-\frac{\beta}{\alpha}}\, y' = 0$$

either *imaginary* if α and β are of the same sign, or *real* if α and β are of opposite signs.

B_2: $p = 0$, $q \neq 0$. The surface is a *cylinder* either *imaginary* if α, β and q are of the same sign, or *real* if there are coefficients with different signs. In particular, if α and β are of the same sign, then we have an *elliptic cylinder*, and if α and β have different signs, then we have a *hyperbolic cylinder*.

B_3: $p \neq 0$. *Paraboloids*. Changing to new coordinates

$$x'' = x', \quad y'' = y', \quad z'' = z' + \frac{q}{2p},$$

we reduce the equation of the surface to the form

$$\alpha x''^2 + \beta y''^2 + 2pz'' = 0.$$

The paraboloid is *elliptic* if α and β are of the same sign, and *hyperbolic* if α and β are of different signs.

In Case C we change to new coordinates x', y', z':

$$x' = x, \quad y' = y, \quad z' = z + \frac{a_3}{a_{33}}.$$

Then the equation will take the form

$$\gamma z'^2 + px + qy + r = 0$$

and we may distinguish the following subcases:

C_1: $p = 0$, $q = 0$. *The surface decomposes into a pair of parallel planes*: *imaginary* if γ and r are of the same sign, or *real* if γ and r have opposite signs, or *coincident* if $r = 0$.

C_2: at least one of the coefficients p or q is non-zero. Preserving the direction of the z-axis, we assume the plane $px + qy + r = 0$ to be the plane $y'z'$. Then the equation will take the form

$$\gamma z'^2 + \delta x' = 0.$$

The surface is a *parabolic cylinder*.

3. Ellipsoid

The equation of the ellipsoid is (Fig. 67)

$$\alpha x^2 + \beta y^2 + \gamma z^2 + \delta = 0.$$

Dividing it by δ and taking $\delta/\alpha = -a^2$, $\delta/\beta = -b^2$, $\delta/\gamma = -c^2$, we reduce it to the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0, \tag{*}$$

where a, b, c are the *semi-axes* of the ellipsoid.

It is seen from equation (*) that the coordinate planes are the planes of symmetry of the ellipsoid, and the origin is the centre of symmetry.

Just as the ellipse is obtained from the circle by uniform compression, so any ellipsoid can be generated by uniformly compressing a

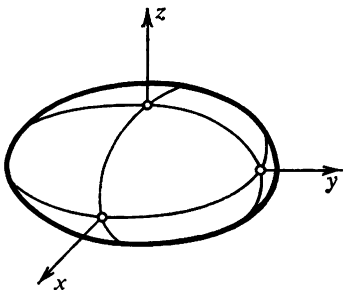

Fig. 67

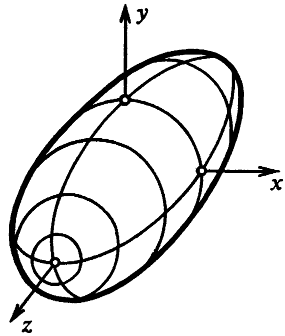

Fig. 68

sphere with respect to two mutually perpendicular planes. Namely, if a is the greatest semi-axis of the ellipsoid, then it can be obtained from the sphere

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{a^2}-1=0$$

by uniformly compressing it with respect to the xy-plane with the compression ratio c/a and with respect to the xz-plane with the compression ratio b/a.

If two semi-axes of an ellipsoid are equal, for instance, $a = b$, then it is called an *ellipsoid of revolution.*

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{c^2}-1=0.$$

Intersecting it with any plane $z = h$ parallel to the xy-plane, we obtain a circle

$$x^2+y^2=\left(1-\frac{h^2}{c^2}\right)a^2, \quad z=h$$

with centre on the z-axis. Hence, in this case the *ellipsoid is generated by revolving the ellipse*

$$\frac{x^2}{a^2}+\frac{z^2}{c^2}-1=0,$$

contained in the xz-plane, about the z-axis (Fig. 68).

If all the three semi-axes of the ellipsoid are equal, then it is a sphere.

The line of intersection (the trace) of an ellipsoid with an arbitrary plane is an ellipse.

Indeed, the trace is a second-order curve. Since this trace is finite (the ellipsoid is finite), it cannot be either a hyperbola or a parabola. Nor can it be a pair of straight lines, and consequently it is an ellipse.

4. Hyperboloids

Just as in the case of the ellipsoid, the equation of hyperboloids can be reduced to the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=0$$

(a *hyperboloid of one sheet*, Fig. 69),

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}+1=0$$

(a *hyperboloid of two sheets*, Fig. 70).

In both hyperboloids the coordinate planes serve as the *planes of symmetry*, and the origin of coordinates as the *centre of symmetry*,

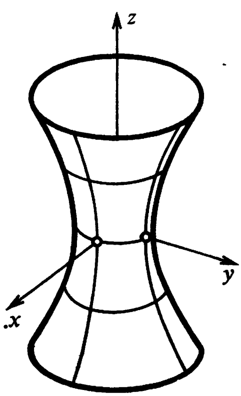

Fig. 69

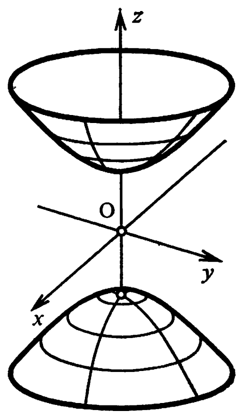

Fig. 70

If the semi-axes a and b of a hyperboloid are equal, then it is called a *hyperboloid of revolution* and is obtained by revolving the hyperbola about the z-axis

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}-1=0, \quad y=0$$

in the case of a hyperboloid of one sheet and the hyperbola

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}+1=0, \quad y=0$$

in the case of a hyperboloid of two sheets.

A general type hyperboloid ($a \neq b$) can be obtained from a hyperboloid of revolution ($a = b$) by uniformly compressing (or stretching) the latter with respect to the xz-plane in the ratio b/a. On crossing hyperboloids with an arbitrary plane various conic sections may result. For instance, the planes $z = h$ parallel to the xy-plane intersect a hyperboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=0$$

in ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{h^2}{c^2}-1=0, \quad z=h,$$

and the planes $y=h$ ($|h| \neq b$) parallel to the xz-plane in hyperbolas

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}-1+\frac{h^2}{b^2}=0, \quad y=h.$$

The plane $y = b$ intersects the hyperboloid along two straight lines:

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=0, \quad y=b.$$

5. Paraboloids

The equations of paraboloids are reduced to the form

$$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

(*an elliptic paraboloid*, Fig. 71),

$$z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

(a *hyperbolic paraboloid*, Fig. 72).

The xz- and yz-planes are the planes of symmetry of paraboloids. Their intersection (the z-axis) is called the *axis of a paraboloid*, and the intersection of its axis with the surface is termed the *vertex*.

If $a = b$ an elliptic paraboloid is said to be a *paraboloid of revolution*. It is formed by revolving a parabola

$$z=\frac{x^2}{a^2}, \qquad y=0$$

about the z-axis. A general-type elliptic paraboloid can be obtained from a paraboloid of revolution

$$z=\frac{x^2}{a^2}+\frac{y^2}{a^2}$$

by uniformly compressing (stretching) it with respect to the xz-plane.

Both paraboloids (elliptic and hyperbolic) are intersected by planes parallel to the xz- and yz-planes along parabolas that are

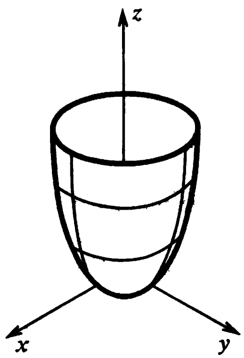

Fig. 71

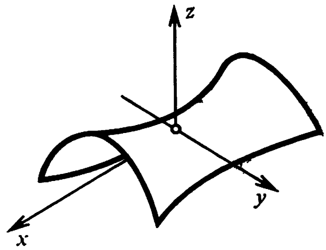

Fig. 72

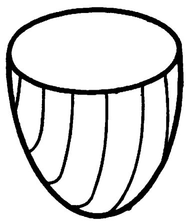

Fig. 73

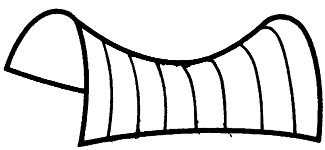

Fig. 74

parallel and equal. Indeed, the planes $x = h$ intersect an elliptic paraboloid along parabolas

$$z - \frac{h^2}{a^2} = \frac{y^2}{b^2}, \quad x = h.$$

If each of these parabolas is displaced in the direction of z, by a line segment h^2/a^2, then we obtain one and the same parabola

$$z = \frac{y^2}{b^2}, \quad x = h.$$

Whence it follows that an *elliptic paraboloid is generated by translating a parabola* $z = \frac{y^2}{b^2}$, $x = 0$, *with its vertex moving along a parabola* $z = \frac{x^2}{a^2}$, $y = 0$ (Fig. 73).

A hyperbolic paraboloid is generated in a similar way (Fig. 74).

The planes parallel to the xy-plane, except for xy-plane itself, cut an elliptic paraboloid along ellipses, and a hyperbolic paraboloid along hyperbolas. The xy-plane intersects a hyperbolic paraboloid along two straight lines.

6. Cone and Cylinders

The equations of the cone and cylinders of the second order may be written in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0 \quad \text{(a \textit{cone}, Fig. 75)},$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0 \quad \text{(an \textit{elliptic cylinder}, Fig. 76)},$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0 \quad \text{(a \textit{hyperbolic cylinder}, Fig. 77)},$$

$$\frac{x^2}{a^2}-py=0 \quad \text{(a \textit{parabolic cylinder}, Fig. 78)}.$$

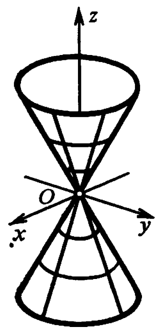

Fig. 75

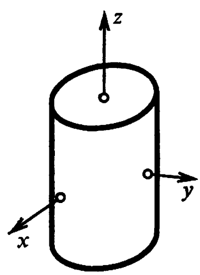

Fig. 76

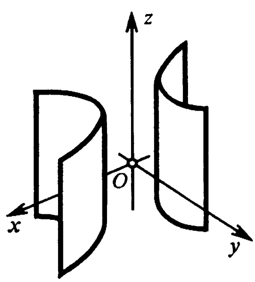

Fig. 77

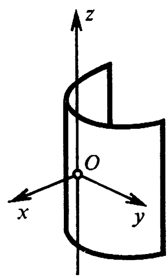

Fig. 78

An arbitrary cone is obtained from a circular cone

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{c^2}=0$$

by compressing (stretching) it uniformly with respect to the xz-plane.

Elliptic, hyperbolic and parabolic cylinders intersect the xy-plane along an ellipse, hyperbola and parabola, respectively, and are generated by straight lines parallel to the z-axis, which intersect the curves mentioned.

An arbitrary elliptic cylinder is obtained from a circular cylinder by compressing (stretching) the latter uniformly with respect to the xz-plane.

We conclude that the cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0,$$

which is called the *asymptotic cone*, is related with the hyperboloids of one and two sheets

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}\pm 1=0$$

in a natural way.

Each plane passing through the z-axis intersects the hyperboloids along hyperbolas, and the cone along two generators which are the asymptotes of these hyperbolas. In particular, the xz-plane ($y = 0$) intersects the hyperboloids along hyperbolas

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}\pm 1=0,$$

and the cone along two straight lines

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=0,$$

which are the asymptotes of these hyperbolas.

7. Rectilinear Generators on Quadric Surfaces

Cones and cylinders are not the only surfaces described by the second-degree equations which contain rectilinear generators. A hyperboloid of one sheet and a hyperbolic paraboloid possess this property as well.

Indeed, *a straight line g_λ, specified by the equations*

$$z=\lambda\left(\frac{x}{a}+\frac{y}{b}\right),\quad 1=\frac{1}{\lambda}\left(\frac{x}{a}-\frac{y}{b}\right), \qquad (*)$$

lies on a hyperbolic paraboloid

$$z=\frac{x^2}{a^2}-\frac{y^2}{b^2} \qquad (**)$$

since any point (x, y, z) satisfying equation $(*)$ also satisfies equation $(**)$ which results as a corollary by termwise multiplication.

In addition to a family g_λ, one more family of straight lines g'_λ is located on a hyperbolic paraboloid:

$$z=\lambda\left(\frac{x}{a}-\frac{y}{b}\right),\quad 1=\frac{1}{\lambda}\left(\frac{x}{a}+\frac{y}{b}\right).$$

Analogously: *on the hyperboloid of one sheet*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=0$$

there are two families of rectilinear generators

$$g_\lambda:\ \frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right),\quad \frac{x}{a}+\frac{z}{c}=\frac{1}{\lambda}\left(1+\frac{y}{b}\right);$$

$$g'_\lambda:\ \frac{x}{a}-\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right),\quad \frac{x}{a}+\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right).$$

In both cases (a hyperbolic paraboloid and hyperboloid of one sheet) *rectilinear generators belonging to one family do not intersect, whereas those belonging to different families intersect.*

The presence of rectilinear generators on a hyperbolic paraboloid and a hyperboloid of one sheet makes it possible to introduce a new

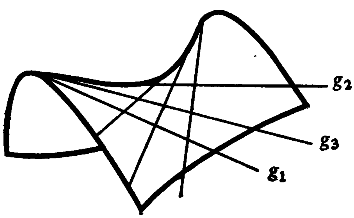

Fig. 79

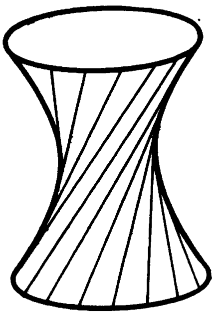

Fig. 80

method of generating these surfaces. Namely, let us take three rectilinear generators g_1, g_2, g_3 belonging to one family. Then each rectilinear generator g belonging to the second family intersects g_1, g_2, g_3. Consequently, the surface is generated by the straight lines g which intersect the three given lines (Fig. 79).

As to the hyperboloid of revolution of one sheet, it is also formed by revolving any of its rectilinear generators about the axis of the surface (Fig. 80).

8. Diameters and Diametral Planes of a Quadric Surface

A straight line, as a rule, intersects a quadric surface at two points. If there are two points of intersection, then the line segment with the end-points being the points of intersection is called the *chord*.

The midpoints of parallel chords of a quadric surface lie in a plane (termed the *diametral plane*). Let us prove this.

Let a quadric surface be defined by the equation in an arbitrary rectangular Cartesian coordinate system

$$a_{11}x^2 + 2a_{12}xy + \ldots + a_{44} = 0. \qquad (*)$$

To simplify notation we will introduce the following:

$$2F = a_{11}x^2 + 2a_{12}xy + \ldots + a_{44},$$
$$F_x = a_{11}x + a_{12}y + a_{13}z + a_{14},$$
$$F_y = a_{21}x + a_{22}y + a_{23}z + a_{24},$$
$$F_z = a_{31}x + a_{32}y + a_{33}z + a_{34}.$$

Let the chords be parallel to the line $\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu}$ and let x, y, z denote the coordinates of the midpoints of an arbitrary chord. Then the coordinates of the end-points of the chord may be written as

$$x_1 = x + \lambda t, \quad y_1 = y + \mu t, \quad z_1 = z + \nu t,$$
$$x_2 = x - \lambda t, \quad y_2 = y - \mu t, \quad z_2 = z - \nu t.$$

Substituting these coordinates into (*) gives

$$2F \pm 2t(\lambda F_x + \mu F_y + \nu F_z)$$
$$+ t^2(a_{11}\lambda^2 + a_{22}\mu^2 + a_{33}\nu^2 + 2a_{12}\lambda\mu + 2a_{23}\mu\nu + 2a_{31}\nu\lambda) = 0.$$

It follows from this that the coefficient of t must be zero

$$\lambda F_x + \mu F_y + \nu F_z = 0. \qquad (**)$$

This is the equation of the diametral plane that corresponds to the chords of the given direction $\lambda : \mu : \nu$.

If a surface has a centre, then each diametral plane passes through the centre. Accordingly, the *centre of a surface is given by*

$$F_x = 0, \quad F_y = 0, \quad F_z = 0. \qquad (***)$$

For a second-degree curve we reason along the same lines. We will only provide the final result.

Let a curve be given by

$$2\Phi = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0.$$

We set

$$\Phi_x = a_{11}x + a_{12}y + a_{13},$$
$$\Phi_y = a_{21}x + a_{22}y + a_{23}.$$

The diameter corresponding to the chords of the direction $\lambda : \mu$, *i.e. parallel to the straight line*

$$\frac{x}{\lambda} = \frac{y}{\mu}$$

is given by

$$\lambda\Phi_x + \mu\Phi_y = 0.$$

The centre of the curve (if any) is found from the simultaneous equations

$$\Phi_x = 0, \quad \Phi_y = 0.$$

9. Axes of Symmetry for a Curve. Planes of Symmetry for a Surface

We now proceed to find the planes of symmetry for a surface defined in arbitrary coordinates.

Let $\lambda : \mu : \nu$ be the direction perpendicular to the symmetry plane. Since the midpoints of chords of direction $\lambda : \mu : \nu$ lie in the plane of symmetry, the latter is given by

$$\lambda F_x + \mu F_y + \nu F_z = 0. \qquad (*)$$

Since the direction $\lambda : \mu : \nu$ is perpendicular to the plane (*), then

$$\frac{a_{11}\lambda + a_{21}\mu + a_{22}\nu}{\lambda} = \frac{a_{21}\lambda + a_{22}\mu + a_{23}\nu}{\mu} = \frac{a_{31}\lambda + a_{32}\mu + a_{33}\nu}{\nu}. \qquad (**)$$

Having found from this system $\lambda : \mu : \nu$ and substituted into (*), we obtain the equation of the plane of symmetry of the surface.

To simplify the finding of $\lambda : \mu : \nu$ from (**), we denote by ξ the common value of the three relations (**). The result will be the equivalent system

$$\left.\begin{aligned} (a_{11}-\xi)\lambda + a_{12}\mu + a_{13}\nu &= 0, \\ a_{21}\lambda + (a_{22}-\xi)\mu + a_{23}\nu &= 0, \\ a_{31}\lambda + a_{32}\mu + (a_{33}-\xi)\nu &= 0. \end{aligned}\right\} \qquad (***)$$

Since λ, μ, ν are not all zero, then

$$\begin{vmatrix} a_{11}-\xi & a_{12} & a_{13} \\ a_{21} & a_{22}-\xi & a_{23} \\ a_{31} & a_{32} & a_{33}-\xi \end{vmatrix} = 0.$$

Finding ξ and substituting the result into (***), we find $\lambda : \mu : \nu$.

If we can find the planes of symmetry of a surface, we can readily find a coordinate system in which the equation of the surface will have canonical form.

So considering second-degree curves similarly we come to the conclusion that for them the *axes of symmetry are given by*

$$\lambda\Phi_x - \mu\Phi_y = 0.$$

From the system

$$\left.\begin{aligned} (a_{11}-\xi)\lambda + a_{12}\mu &= 0, \\ a_{21}\lambda + (a_{22}-\xi)\mu &= 0. \end{aligned}\right\}$$

where ξ is a root of the equation

$$\begin{vmatrix} a_{11}-\xi & a_{12} \\ a_{21} & a_{22}-\xi \end{vmatrix} = 0,$$

we find $\lambda : \mu$.

The system of coordinates, in which the equation of the curve assumes canonical form, is determined from the considerations similar to those used above for surfaces.

EXERCISES TO CHAPTER VII

1. A curve in the plane

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a = 0$$

is an ellipse (hyperbola, parabola). What does the quadric surface

$$z = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a$$

represent?

2. Show that the quadric surface

$$\lambda (a_1x + b_1y + c_1z + d_1)^2 + \mu (a_2x + b_2y + c_2z + d_2)^2 = 0$$

is divided into a pair of planes.

3. To obtain the projection on the xy-plane of the curve of intersection of the surface

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + \ldots + a_{44} = 0 \qquad (*)$$

with the plane

$$z = ax + by + c,$$

we should substitute $z = ax + by + c$ in equation (*). Prove this.

4. Show that the sections of a quadric surface by parallel planes are homothetic and are positioned similarly.

5. Show that a conical surface generated by straight lines passing through a given point and intersecting a second-degree curve is a quadric surface.

6. Form the equation of the surface generated by the straight line

$$\left.\begin{array}{l} z = ax + b, \\ z = cy + d \end{array}\right\} \quad (a, b, c, d \neq 0)$$

rotating about the z-axis.

7. If $a < c$, then the ellipsoid of revolution

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1,$$

is a locus of points the sum of whose distances from the two given points (the foci) is constant. Find the foci of the ellipsoid.

8. Suppose we have an ellipsoid

$$\alpha x^2 + \beta y^2 + \gamma z^2 + \delta = 0.$$

Show that if the surface

$$\alpha x^2 + \beta y^2 + \gamma z^2 + \delta - \lambda (x^2 + y^2 + z^2 + \mu) = 0$$

decomposes into a pair of planes, then these planes intersect the ellipsoid along circles. Use this fact to justify the method of finding circular sections of the ellipsoid.

9. Show that the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

may be specified by the parametric equations:

$$x = a \cos u \cos v, \quad y = b \cos u \sin v, \quad z = c \sin u.$$

10. What is the surface

$$(a_1x + b_1y + c_1z)^2 + (a_2x + b_2y + c_2z)^2 + (a_3x + b_3y + c_3z)^2 = 1,$$

if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0?$$

11. Find the circular sections of the hyperboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0.$$

12. Show that through any point in space not belonging to the coordinate planes, pass three surfaces of the family

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1$$

(λ, the parameter): an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets.

13. Show that the plane $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} + \frac{z+z_0}{2} = 0$ passing through the point (x_0, y_0, z_0) of a hyperbolic paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} + z = 0$ intersects the paraboloid along two rectilinear generators belonging to different families.

14. Find rectilinear generators of a hyperbolic paraboloid $z = axy$.

15. Form the equation of a surface generated by straight lines parallel to the xy-plane and intersecting two given skew lines.

16. Show that the equation of a circular cone with the vertex at the origin, the axis $\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}$, and the vertex angle 2α can be written in the form

$$\frac{(\lambda x+\mu y+\nu z)^2}{(x^2+y^2+z^2)(\lambda^2+\mu^2+\nu^2)}=\cos^2\alpha.$$

17. Show that the equation of a circular cylinder of radius R and with the axis $\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}$ can be written in the form

$$x^2+y^2+z^2-R^2=\frac{(\lambda x+\mu y+\nu z)^2}{\lambda^2+\mu^2+\nu^2}.$$

18. Find the axis of the circular cone

$$x^2+y^2+z^2-(ax+by+cz)^2=0.$$

19. Find the vertex and the axis of the parabola

$$(ax+by+c)^2+\alpha x+\beta y+\gamma=0.$$

Part Two

DIFFERENTIAL GEOMETRY

Chapter VIII

TANGENT AND OSCULATING PLANES OF CURVE

1. Concept of Curve

The concept of transformation of a figure, or a set of points, is known from elementary geometry. Let us recall it. If each point of a figure F is displaced somehow, then we obtain a new figure F' which is said to be obtained by a *transformation* from F. A transformation of F is said to be *continuous* if it sends near points of F to the near points of F', which means that if a point X of F is sent into point X' of F', then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that any point Y of F, which is from X at a distance less than δ, is carried into a point of F', which is from X' at a distance less than ε. A transformation sending different points of a figure F into different

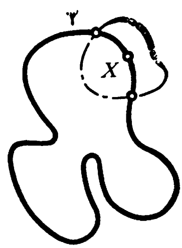

Fig. 81

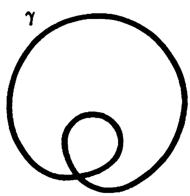

Fig. 82

points of a figure F' is said to be *topological* if it is continuous as well as its converse of F' into F. A transformation of a figure is said to be *locally topological* if it is topological in a sufficiently small neighbourhood of each of its points.

We now give several definitions related to the concept of curve. We will call a figure obtained by a topological transformation of an open line segment an *elementary curve*. A figure whose each point possesses a three-dimensional neighbourhood such that the part of the figure contained in it is an elementary curve is called a *simple curve* (Fig. 81). A figure obtained by a locally topological transformation of a simple curve is known as a *generic curve*. The generic curve in Fig. 82 is obtained by a locally topological transformation of the circumference.

Due to these definitions, the study of any curve "in the small" is reduced to that of an elementary curve. Let γ be an elementary curve which is a topological transformation of a line segment AB. If we introduce a coordinate t on the straight line AB as the number axis, then any transformation of the line segment AB into γ can be given by the equations

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t), \qquad (*)$$

where f_1, f_2 and f_3 are continuous functions, with

$$(f_1(t') - f_1(t''))^2 + (f_2(t') - f_2(t''))^2 + (f_3(t') - f_3(t''))^2 \neq 0$$

for different values of t' and t''.

We call the equations (*) the *parametric equations of the curve* γ, t being the parameter. An elementary curve admits different methods of specifying it parametrically. E.g., γ can be given by the equations

$$x = f_1(\varphi(\tau)), \quad y = f_2(\varphi(\tau)), \quad z = f_3(\varphi(\tau)),$$

where $\varphi(\tau)$ is any continuous, strictly monotonic function of τ

2. Regular Curve

We call a curve γ *regular* (i.e., k times differentiable) if it admits a regular parametrization, or specification by parametric equations

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t),$$

where f_1, f_2 and f_3 are regular functions (i.e., k times differentiable) which satisfy the condition

$$f_1'^2 + f_2'^2 + f_3'^2 \neq 0.$$

For $k = 1$, a curve is said to be *smooth*.

A curve is said to be *analytic* if it admits an analytic parametrization (i.e., the functions f_1, f_2 and f_3 are analytic).

Certain curves admit a parametric representation

$$x = t, \quad y = \varphi(t), \quad z = \psi(t),$$

or, which is equivalent,

$$y = \varphi(x), \quad z = \psi(x),$$

for a suitable choice of the coordinate axes. This parametrization sometimes turns out to be very convenient in the study of curves. Accordingly, the question arises, when does a curve admit such a parametrization at least "in the small"?

The answer is supplied by the following theorem.

Let γ be a regular curve, and

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t)$$

its regular parametric representation in the neighbourhood of the point (x_0, y_0, z_0) *corresponding to* $t = t_0$. *If* $t = t_0$, $f_1'(t) \neq 0$, *then the curve can be given by the equations*

$$y = \varphi(x), \quad z = \psi(x)$$

in a sufficiently small neighbourhood of (x_0, y_0, z_0), *where* $\varphi(x)$ *and* $\psi(x)$ *are regular functions of* x.

Proof. By the implicit function theorem, there exists a regular function $\chi(x)$ equal to t_0 for $x = x_0$, identically satisfying the equation

$$x = f_1(\chi(x))$$

for x near to x_0. Differentiating the identity for $x = x_0$, we obtain $1 = f_1'(t_0)\chi'(x_0)$. Hence, $\chi'(x_0) \neq 0$, which means that the function $\chi'(x)$ is monotonic in the vicinity of $x = x_0$, and we can introduce the parameter x instead of t by putting $t = \chi(x)$.

We obtain

$$y = f_2(\chi(x)), \quad z = f_3(\chi(x)).$$

Q.E.D.

3. Singular Points of a Curve

Let γ be a curve, and P a point in it. P is called an *ordinary point* if the curve admits a smooth parametrization

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t), \quad f_1'^2 + f_2'^2 + f_3'^2 \neq 0$$

in its neighbourhood. If there is no such parametrization, then the point is said to be *singular*. The problem of singular points of a plane curve is in many practically important cases solved by the following theorem.

Let γ *be a curve given by parametric equations*

$$x = f_1(t), \quad y = f_2(t).$$

Then a point P *of the curve is ordinary if the derivative of the functions* f_1 *and* f_2, *which is the first non-zero one, is odd at it.* P *is singular if its derivative which is the first non-zero one is even.*

Proof. Without loss of generality, we assume that P is at the origin, and the value of the parameter t, associated with P, is zero. By the Taylor formula,

$$x = \frac{t^n}{n!}\left(f_1^{(n)}(0) + \varepsilon_1(t)\right), \quad y = \frac{t^m}{m!}\left(f_2^{(m)}(0) + \varepsilon_2(t)\right).$$

For definiteness, let $n \leqslant m$.

In the case of odd n, we introduce the parameter

$$\tau = t^n$$

instead of t, τ being a monotonic function of t. The obtained parametrization of the curve is smooth, since

$$\left.\frac{df_1}{d\tau}\right|_P = \lim_{t\to 0} \frac{f_1(t)}{t^n} = \frac{1}{n!} f_1^{(n)}(0) \neq 0.$$

Therefore, P is ordinary for odd n.

Now, let n be even, in which case $f_1(t)$ does not change sign (of $f_1^{(n)}(0)$) in the neighbourhood of P. Therefore, the curve is either in

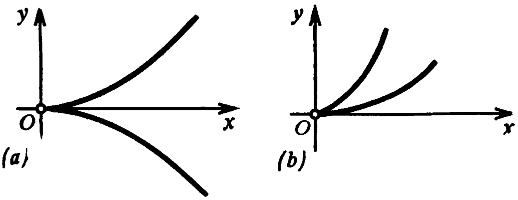

Fig. 83

the half-plane $x > 0$ if $f_1^{(n)}(0) > 0$, or the half-plane $x < 0$ if $f_1^{(n)}(0) < 0$ in the neighbourhood of P. Assume that P is ordinary. The curve then admits a smooth parametrization

$$x = \varphi_1(\tau), \quad y = \varphi_2(\tau), \quad \varphi_1'^2 + \varphi_2'^2 \neq 0$$

in the neighbourhood. Since $\varphi_1'^2 + \varphi_2'^2 \neq 0$, and φ_1 is of the order not higher than φ_2, $\varphi_1' \neq 0$ at P. Hence, $\varphi_1(\tau)$ in the neighbourhood of P changes sign; therefore, the curve γ is placed in both half-planes $x > 0$ and $x < 0$ in the neighbourhood of P, and we have come to a contradiction. Thus, the point P is singular for even n.

For even n and odd m, a singular point is called a *cusp of the first kind.* The form of the curve in the neighbourhood of such a point is shown in Fig. 83*a*. For even n and even m, $n < m$, a singular point is called a *cusp of the second kind.* The form of the curve of such a singular point in its neighbourhood is shown in Fig. 83*b*.

The case $m = n$ is reduced to the above ($n < m$) with a corresponding rotation of the coordinate axes.

4. Vector Function of Scalar Argument

Below, we will often resort to the elementary means of vector analysis, due to which we give principal definitions.

A *vector function* is said to be given in an interval $a < t < b$ if each value t is associated with a vector $\mathbf{f}(t)$. The concept of limit is introduced for vector functions in the same manner as for scalar ones. Viz., the *limit of a vector function* $\mathbf{f}(t)$ as $t \to t_0$ is a vector $\mathbf{c}$ such that

$$\lim_{t\to t_0} |\mathbf{f}(t) - \mathbf{c}| = 0.$$

*858. $\begin{vmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ n & 1 & 2 & \dots & n-2 & n-1 \\ n-1 & n & 1 & \dots & n-3 & n-2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & 3 & 4 & \dots & n & 1 \end{vmatrix}$.

859. $\begin{vmatrix} a & a+h & a+2h & \dots & a+(n-2)h & a+(n-1)h \\ a+(n-1)h & a & a+h & \dots & a+(n-3)h & a+(n-2)h \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a+h & a+2h & a+3h & \dots & a+(n-1)h & a \end{vmatrix}$.

*860. $\begin{vmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon & \varepsilon^2 & \varepsilon^3 & \dots & \varepsilon^{n-1} \\ 1 & \varepsilon^2 & \varepsilon^4 & \varepsilon^6 & \dots & \varepsilon^{2(n-1)} \\ 1 & \varepsilon^3 & \varepsilon^6 & \varepsilon^9 & \dots & \varepsilon^{3(n-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \varepsilon^{n-1} & \varepsilon^{2(n-1)} & \varepsilon^{3(n-1)} & \dots & \varepsilon^{(n-1)^2} \end{vmatrix}$,

where $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

Solve the following matrix equations:

861. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot X = \begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix}$.

862. $X \cdot \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -5 & 6 \end{pmatrix}$.

863. $\begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix} \cdot X \cdot \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 14 & 16 \\ 9 & 10 \end{pmatrix}$.

864. $\begin{pmatrix} 1 & 2 & -3 \\ 3 & 2 & -4 \\ 2 & -1 & 0 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & -3 & 0 \\ 10 & 2 & 7 \\ 10 & 7 & 8 \end{pmatrix}$.

865. $X \cdot \begin{pmatrix} 5 & 3 & 1 \\ 1 & -3 & -2 \\ -5 & 2 & 1 \end{pmatrix} = \begin{pmatrix} -8 & 3 & 0 \\ -5 & 9 & 0 \\ -2 & 15 & 0 \end{pmatrix}$.

866. $\begin{pmatrix} 2 & -3 & 1 \\ 4 & -5 & 2 \\ 5 & -7 & 3 \end{pmatrix} \cdot X \cdot \begin{pmatrix} 9 & 7 & 6 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -2 \\ 18 & 12 & 9 \\ 23 & 15 & 11 \end{pmatrix}$.

867. $\begin{pmatrix} 2 & -3 \\ 4 & -6 \end{pmatrix} \cdot X = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$.

868. $X \cdot \begin{pmatrix} 3 & 6 \\ 4 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 9 & 18 \end{pmatrix}$.

A vector function with continuous derivatives up to order k on an interval (a, b) is said to be *k times differentiable* on the interval.

Let $\mathbf{f}(t)$ be a vector function, and $\lambda(t)$, $\mu(t)$ and $\nu(t)$ the components of the vector $\mathbf{f}(t)$. If the scalar functions λ, μ and ν are differentiable, then the vector function $\mathbf{f}$ is also differentiable. Conversely, if $\mathbf{f}$ is differentiable, then λ, μ and ν are differentiable.

In fact, if we denote the base vectors by $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, then

$$\mathbf{f}(t) = \lambda(t)\,\mathbf{e}_1 + \mu(t)\,\mathbf{e}_2 + \nu(t)\,\mathbf{e}_3. \qquad (*)$$

It is obvious that the differentiability of the vector function $\mathbf{f}$ follows from that of the functions λ, μ and ν. To prove the converse statement, it suffices to notice that

$$\lambda(t) = \mathbf{f}(t)\,\mathbf{e}_1, \quad \mu(t) = \mathbf{f}(t)\,\mathbf{e}_2, \quad \nu(t) = \mathbf{f}(t)\,\mathbf{e}_3.$$

The Taylor formula is also valid for vector functions. Viz.,

$$\mathbf{f}(t+h) = \mathbf{f}(t) + h\mathbf{f}'(t) + \ldots + \frac{h^n}{n!}\left(\mathbf{f}^{(n)}(t) + \varepsilon(t, h)\right),$$

where $|\varepsilon(t, h)| \to 0$ as $h \to 0$.

For proof, it suffices to represent the vector function $\mathbf{f}(t)$ in the form $(*)$, and apply the Taylor formula to the functions $\lambda(t)$, $\mu(t)$ and $\nu(t)$.

The three equations for specifying a curve parametrically

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t)$$

can be represented as one vector equation

$$\mathbf{r} = \mathbf{f}(t), \qquad (**)$$

where $\mathbf{r}$ is the vector of a point on the curve, i.e., whose origin is at the origin of coordinates, and the end-point on the curve

$$\mathbf{f}(t) = f_1(t)\,\mathbf{e}_1 + f_2(t)\,\mathbf{e}_2 + f_3(t)\,\mathbf{e}_3.$$

The equation $(**)$ is called the *vector equation of the curve*. The regularity of the curve means that of the vector function $\mathbf{f}$, whereas the condition $f_1'^2 + f_2'^2 + f_3'^2 \neq 0$ means that the vector $\mathbf{f}' \neq 0$.

5. Tangent to a Curve

The concept of tangent to a curve is already known to us. Now, we give another definition equivalent to the prior one, but more convenient for our immediate goals.

Let γ be a curve, P a point on it, and g a straight line passing through P. Take a point Q near to P, and denote its distances from P and g by d and δ, respectively (Fig. 84). We call the straight line g a *tangent* to γ at P if $\delta/d \to 0$ as $Q \to P$.

If γ possesses a tangent at P, then the straight line PQ tends to the tangent as $Q \to P$. Conversely, if a straight line PQ approaches a certain straight line as $Q \to P$, then it is a tangent, to prove which it suffices to notice that δ/d is the sine of the angle made by g and PQ.

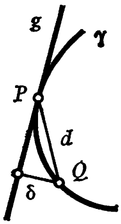

Fig. 84

A smooth curve γ has one, and only one, tangent at each point. If

$$\mathbf{r} = \mathbf{f}(t)$$

is the vector equation of the curve, then the tangent at the point P corresponding to a value of parameter t has the same direction as the vector $\mathbf{f}'(t)$.

Proof. Assume that γ has a tangent g at the point $P(t)$. Let $\boldsymbol{\tau}$ be the unit vector associated with the straight line g. The distance d from the point $Q(t+h)$ to P equals $|\mathbf{f}(t+h) - \mathbf{f}(t)|$, whereas the distance δ from Q to the tangent is $|(\mathbf{f}(t+h) - \mathbf{f}(t)) \wedge \boldsymbol{\tau}|$.

By definition of a tangent,

$$\frac{\delta}{d} = \frac{|(\mathbf{f}(t+h)-\mathbf{f}(t)) \wedge \boldsymbol{\tau}|}{|\mathbf{f}(t+h)-\mathbf{f}(t)|} \to 0 \text{ as } h \to 0.$$

However,

$$\frac{|(\mathbf{f}(t+h)-\mathbf{f}(t)) \wedge \boldsymbol{\tau}|}{|\mathbf{f}(t+h)-\mathbf{f}(t)|} = \frac{\left|\frac{(\mathbf{f}(t+h)-\mathbf{f}(t)) \wedge \boldsymbol{\tau}}{h}\right|}{\left|\frac{\mathbf{f}(t+h)-\mathbf{f}(t)}{h}\right|} \to \frac{|\mathbf{f}'(t) \wedge \boldsymbol{\tau}|}{|\mathbf{f}'(t)|}.$$

Hence, $\mathbf{f}' \wedge \boldsymbol{\tau} = 0$. Because $\mathbf{f}' \neq 0$, the vectors $\mathbf{f}'$ and $\boldsymbol{\tau}$ are collinear. Thus, if a tangent does exist, then it has the direction of the vector $\mathbf{f}'$, and, is, therefore, unique.

That a straight line g passing through the point P, and having the direction of the vector $\mathbf{f}'$, is a tangent, is also true. Indeed, the above argument shows that

$$\frac{\delta}{d} = \frac{\left|(\mathbf{f}(t+h)-\mathbf{f}(t)) \wedge \frac{\mathbf{f}'(t)}{|\mathbf{f}'(t)|}\right|}{|\mathbf{f}(t+h)-\mathbf{f}(t)|} \to \frac{|\mathbf{f}'(t) \wedge \mathbf{f}'(t)|}{|\mathbf{f}'(t)|^2} = 0$$

for such a curve.

6. Equations of Tangents for Various Methods of Specifying a Curve

As we know, a straight line passing through a point (x_0, y_0, z_0), and having the direction of a vector $\overrightarrow{(a, b, c)}$, can be given by the equations

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}.$$

Since we know the direction of the tangent to the curve, we can easily derive its equation. If a curve is given by

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t),$$

then the tangent at the point associated with a value of the parameter t has the same direction as the vector $\mathbf{f}'(t)$ with components $f_1'(t)$, $f_2'(t)$, $f_3'(t)$. Therefore, the equations of the tangent at this point are

$$\frac{x-f_1(t)}{f_1'(t)} = \frac{y-f_2(t)}{f_2'(t)} = \frac{z-f_3(t)}{f_3'(t)}.$$

In the case of a plane curve given by the equations

$$x = f_1(t), \quad y = f_2(t),$$

that of the tangent is written as

$$\frac{x-f_1(t)}{f_1'(t)} = \frac{y-f_2(t)}{f_2'(t)}.$$

If a curve is given by the equations

$$y = f(x), \quad z = \varphi(x), \qquad (*)$$

then the equations of the tangent are obtained simply from that of a parametrically given curve. It suffices to notice that the specification of a curve by the equations (*) is equivalent to the parametric representation

$$x = t, \quad y = f(t), \quad z = \varphi(t).$$

Therefore, the equations of the tangent to the curve at a point with abscissa x_0 are

$$x - x_0 = \frac{y-f(x_0)}{f'(x_0)} = \frac{z-\varphi(x_0)}{\varphi'(x_0)},$$

or, in equivalent form,

$$y = f(x_0) + f'(x_0)(x - x_0),$$
$$z = \varphi(x_0) + \varphi'(x_0)(x - x_0).$$

If the curve is plane, and given by $y = f(x)$, then we obtain the familiar equation

$$y = f(x_0) + f'(x_0)(x - x_0).$$

We now make up the equations of the tangent to a curve given in implicit form

$$\varphi(x, y, z) = 0, \quad \psi(x, y, z) = 0$$

at a point (x_0, y_0, z_0), where the rank of the matrix

$$\begin{pmatrix} \varphi_x & \varphi_y & \varphi_z \\ \psi_x & \psi_y & \psi_z \end{pmatrix}$$

is two.

Let

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

be a regular parametrization of the curve in the neighbourhood of (x_0, y_0, z_0), and t_0 the associated value of the parameter.

Differentiating the identities

$$\varphi(x(t), y(t), z(t)) = 0, \quad \psi(x(t), y(t), z(t)) = 0,$$

we obtain

$$\varphi_x x' + \varphi_y y' + \varphi_z z' = 0, \quad \psi_x x' + \psi_y y' + \psi_z z' = 0.$$

Hence, the tangent vector $\mathbf{r}'(x', y', z')$ is perpendicular to the vectors $\overrightarrow{(\varphi_x, \varphi_y, \varphi_z)}$, $\overrightarrow{(\psi_x, \psi_y, \psi_z)}$, and, therefore, has the direction of their vector product. Thus, we come to the equation of the tangent

$$\frac{x - x_0}{\begin{vmatrix} \varphi_y & \varphi_z \\ \psi_y & \psi_z \end{vmatrix}} = \frac{y - y_0}{\begin{vmatrix} \varphi_z & \varphi_x \\ \psi_z & \psi_x \end{vmatrix}} = \frac{z - z_0}{\begin{vmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{vmatrix}},$$

(the derivatives $\varphi_x, \varphi_y, \ldots, \varphi_z$ being taken at the point of contact (x_0, y_0, z_0)).

If the curve is plane, and given by an equation $\varphi(x, y) = 0$, then the equation of its tangent is

$$\frac{x - x_0}{\varphi_y} = \frac{y - y_0}{-\varphi_x},$$

or

$$(x - x_0)\varphi_x + (y - y_0)\varphi_y = 0,$$

to see which it suffices to notice that, being space, this curve is given by the two equations $\varphi(x, y) = 0$ and $z = 0$.

The plane passing through a point P, and perpendicular to the tangent at this point, is called the *normal plane to the curve* at P. Obviously, it is not hard to form its equation, since the tangent vector is perpendicular to the plane.

7. Osculating Plane of a Curve

Let γ be a curve, P a point on it, and α a plane passing through P. Denote the distance from a point Q on the curve to P by d, and that from Q to α by δ. We will call α the *osculating plane* of γ at P if the ratio $\delta/d^2 \to 0$ as $Q \to P$ (Fig. 85).

A twice differentiable curve γ has an osculating plane at each of its points. Meanwhile, it is either unique, or any plane containing the tangent to the curve is osculating.

If

$$\mathbf{r} = \mathbf{r}(t)$$

is the equation of the curve, then the osculating plane is parallel to the vectors $\mathbf{r}'$ *and* $\mathbf{r}''$.

Proof. Let α be the osculating plane of γ at a point P associated with a value of the parameter t. Denote by $\mathbf{e}$ the unit normal vector. We have

$$d = |\mathbf{r}(t+h) - \mathbf{r}(t)|, \quad \delta = |\mathbf{e}(\mathbf{r}(t+h) - \mathbf{r}(t))|,$$

$$\frac{\delta}{d^2} = \frac{|\mathbf{e}(\mathbf{r}(t+h)-\mathbf{r}(t))|}{(\mathbf{r}(t+h)-\mathbf{r}(t))^2} = \frac{\left|\mathbf{e}\left(\mathbf{r}'(t)h+\frac{\mathbf{r}''(t)}{2}h^2+\varepsilon_1 h^2\right)\right|}{(\mathbf{r}'(t)h+\varepsilon_2 h)^2}$$

$$= \frac{\left|\frac{\mathbf{e}\mathbf{r}'(t)}{h}+\frac{\mathbf{e}\mathbf{r}''(t)}{2}+\varepsilon_1\right|}{\mathbf{r}'^2(t)+\varepsilon_3}.$$

Since $\delta/d^2 \to 0$, $\varepsilon_1 \to 0$, $\varepsilon_3 \to 0$ as $Q \to P$, and $\mathbf{r}'(t) \neq 0$, $\mathbf{e}\mathbf{r}'(t) = 0$, we derive $\mathbf{e}\mathbf{r}''(t) = 0$. Thus, if an osculating plane does exist, then the vectors $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ are parallel to it.

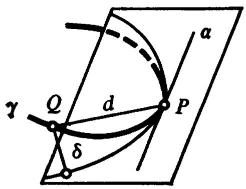

Fig. 85

That an osculating plane always exists can be seen easily, for which we take a plane α parallel to $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$, regarding any plane as parallel to the zero vector. Then $\mathbf{e}\mathbf{r}'(t)=0$, $\mathbf{e}\mathbf{r}''(t) = 0$, and, therefore,

$$\frac{\delta}{d^2} = \frac{|\varepsilon_1|}{\mathbf{r}'^2(t)+\varepsilon_3} \to 0 \text{ as } Q \to P.$$

Thus, there is an osculating plane at each point of the curve. It is unique if $\mathbf{r}'$ and $\mathbf{r}''$ are non-collinear. However, if they are collinear, or $\mathbf{r}'' = 0$, then any plane passing through the tangent to the curve is osculating.

To make up the equation of the osculating plane of a curve at a point P, consider an arbitrary point $A(x, y, z)$ of the plane. Then the three vectors $\overrightarrow{PA}$, $\mathbf{r}'$ and $\mathbf{r}''$ are coplanar, each of which is either parallel to the plane α or is in it. Therefore, their scalar triple product is zero.

Let the curve be given by the equations

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t).$$

Then the coordinates of the vector $\mathbf{r}'$ are f_1', f_2', f_3', those of $\mathbf{r}''$ are f_1'', f_2'', f_3'', and those of $\overrightarrow{PA}$ are $x - f_1$, $y - f_2$, $z - f_3$.

Since the scalar triple product of $\overrightarrow{PA}$, $\mathbf{r}'$ and $\mathbf{r}''$ is zero, the osculating plane equation is

$$\begin{vmatrix} x-f_1(t) & y-f_2(t) & z-f_3(t) \\ f_1'(t) & f_2'(t) & f_3'(t) \\ f_1''(t) & f_2''(t) & f_3''(t) \end{vmatrix} = 0.$$

Each straight line passing through a point of a curve perpendicular to the tangent is called a *normal* to the curve. Two normals can be distinguished in the case where the osculating plane is unique, viz., the *principal normal*, or normal lying in the osculating plane, and the *binormal*, or the normal perpendicular to the osculating plane. With the known tangent and osculating plane equations, the derivation of the principal normal and binormal equations is, obviously, not complicated.

8. Envelope of a Family of Plane Curves

A curve γ: $x = x(t)$, $y = y(t)$ is said to be the *envelope* of a family of curves γ_t: $\varphi(x, y, t) = 0$, t being the parameter of the

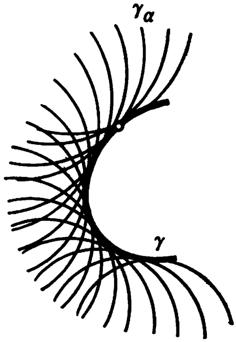

Fig. 86

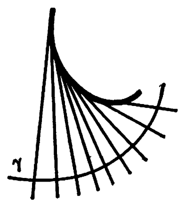

Fig. 87

family, if γ_t touches γ at each of its points (t), i.e., if they possess a common tangent (Fig. 86).

With such a definition of an envelope, on substituting $x(t)$ and $y(t)$ in the equation $\varphi(x, y, t) = 0$, we obtain the identity

$$\varphi(x(t), \quad y(t), t) = 0.$$

Differentiating, we get

$$\varphi_x x' + \varphi_y y' + \varphi_t = 0.$$

It turns out that $\varphi_x x' + \varphi_y y' = 0$. In fact, $\overrightarrow{(x', y')}$ is the tangent vector for the curve γ, whereas $\overrightarrow{(\varphi_y \; - \varphi_x)}$ that for γ_t at the same point. Since they are collinear, $x'/\varphi_y = y'/-\varphi_x$, and $\varphi_x x' + \varphi_y y' = 0$.

Thus, the functions $x(t)$ and $y(t)$ satisfy the simultaneous equations

$$\varphi(x, y, t) = 0, \quad \varphi_t(x, y, t) = 0,$$

and can be found from them.

As an example, we find the envelope of the normals to a plane curve (Fig. 87). Let the curve be given by two equations

$$x = x(t), \quad y = y(t).$$

Since x' and y' are the coordinates of the tangent vector, the equation of the normal is

$$(x - x(t))\, x' + (y - y(t))\, y' = 0. \qquad (*)$$

Differentiating with respect to t, we obtain

$$(x - x(t))\, x''(t) + (y - y(t))\, y''(t) - x'^2(t) - y'^2(t) = 0.$$

Solving it simultaneously with $(*)$ for x and y, we derive the equations of the envelope

$$x = x(t) - \frac{(x'^2 + y'^2)\, y'}{y''x' - x''y'}, \quad y = y(t) - \frac{(x'^2 + y'^2)\, x'}{x''y' - y''x'},$$

assuming that $y''x' - x''y' \neq 0$.

The envelope of the normals to a plane curve is called its *evolute*. The evolute of a curve possesses many remarkable properties, and we note some of them in the sequel.

EXERCISES TO CHAPTER VIII

1. A point M moves in space so that its projection onto the xy-plane moves uniformly along a circle $x^2 + y^2 = a^2$ with angular velocity ω, and the projection onto the z-axis moves uniformly with

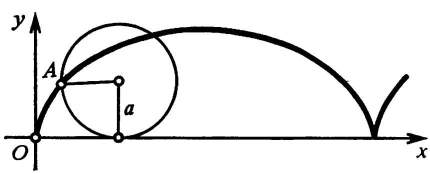

Fig. 88

velocity c. The curve described by M is called a *helix*. Make up its parametric equations, taking time t as the parameter. Assume that M has the coordinates a, 0, 0 at the initial moment $t = 0$.

2. A circle of radius a rolls uniformly without slipping with velocity v along the x-axis. Find the equation of the curve described by a point of the circle if it coincides with the origin at the initial moment $t = 0$ (such a curve is called a *cycloid*, Fig. 88).

3. Find the parametric equation of the curve

$$x^3 + y^3 - 3axy = 0,$$

taking $t = y/x$ as the parameter (*Cartesian folium*; Fig. 89).

4. A helix

$$x = a\cos\omega t, \quad y = a\sin\omega t, \quad z = ct$$

is projected onto the xy-plane by straight lines parallel to the yz-

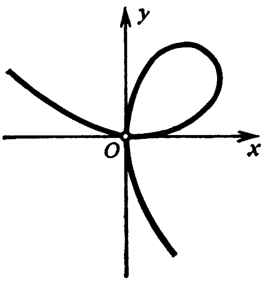

Fig. 89

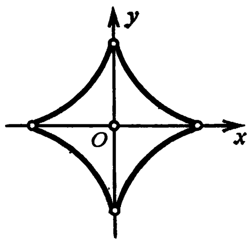

Fig. 90

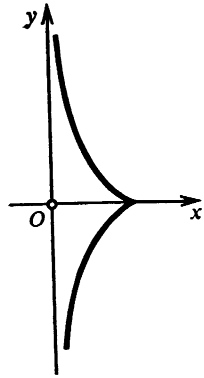

Fig.91

plane, and making an angle θ with the z-axis. For what θ will the projection possess singular points? Clarify their nature.

5. Find the singular points of the cycloid

$$x = vt - a\sin\frac{vt}{a}, \quad y = a\left(1 - \cos\frac{vt}{a}\right),$$

and clarify their nature.

6. Find the singular points of the *astroid*

$$x = a\cos^3 t, \quad y = a\sin^3 t,$$

and clarify their nature (Fig. 90).

7. Find the singular points of the *tractrix*

$$x = a\sin t, \quad y = a\left(\cos t + \ln\tan\frac{t}{2}\right) \quad (0 < t < \pi),$$

and clarify their nature (Fig. 91).

8. Make up the equations of the tangent, osculating plane, normal plane, principal normal and binormal at the point (1, 0, 0) on the helix

$$x = \cos t, \quad y = \sin t, \quad z = t.$$

9. Make up the equation of the tangent to the curve given by the equations

$$x^2 + y^2 + z^2 = 1, \quad x^2 + z^2 = x$$

at the point (0, 0, 1).

10. Find the equation of the tangent to the curve $x = t^2$, $y = t^3$ at the point (0, 0).

11. Find the equation of a parabola of the form

$$y = x^2 + ax + b,$$

which is tangent to the circle,

$$x^2 + y^2 = 2.$$

12. Prove that, for a tractrix, part of the tangent between the point of contact and the y-axis is constant (see Ex. 7), i.e., independent of the choice of a point of tangency.

13. Line segments of the same length are marked off on the binormals to a helix. What is the curve formed by the end-points?

14. At what angle do the hyperbolas

$$xy = c_1, \quad x^2 - y^2 = c_2$$

intersect?

15. Given the family of curves g_λ

$$\frac{x^2}{a^2-\lambda} + \frac{y^2}{b^2-\lambda} = 1,$$

prove that two of them pass through each point of the xy-plane not on the coordinate axes, and that they intersect at right angles.

16. Show that if the tangents to a curve pass through the same point, then the curve is either a straight line or a straight line segment.

17. Show that the tangents to a helix

$$x = a \cos \omega t, \quad y = a \sin \omega t, \quad z = bt$$

make a constant angle with the xy-plane, and that the principal normals intersect the z-axis.

18. Show that if the tangents to a curve are parallel to a certain plane, then the curve is plane.

19. On what condition are the straight lines

$$\left.\begin{aligned} a_1(t)\,x + b_1(t)\,y + c_1(t)\,z + d_1(t) = 0 \\ a_2(t)\,x + b_2(t)\,y + c_2(t)\,z + d_2(t) = 0 \end{aligned}\right\}$$

tangent to a certain curve? Find the curve.

20. Make up the equation of the osculating plane of the curve given by equations

$$\varphi(x, y, z) = 0, \quad \psi(x, y, z) = 0$$

at a point (x, y, z).

21. Given the osculating planes of a curve

$$A(t)x + B(t)y + C(t)z + D(t) = 0,$$

find the equations of the curve

$$x = x(t), \quad y = y(t), \quad z = z(t).$$

22. Find the equation and form of the envelope of the family of straight lines cutting a triangle of area $2a^2$ off the quadrant Oxy.

23. Find the equation and form of the envelope of a family of straight lines intercepting a segment of the same length a on the coordinate axes.

24. Find the envelope of the trajectories of a point particle projected from the origin of coordinates with velocity v_0 at various angles (*parabola of safety*).

Chapter IX

CURVATURE AND TORSION OF CURVE

1. Length of a Curve

Let an elementary curve be given by equations

$$x = x(t), \quad y = y(t), \quad z = z(t).$$

The limit of the lengths of broken lines starting at points $(x(t_i), y(t_i))$, $t_1 = a, t_2, t_3, \ldots, t_n = b$ $(t_1 < t_2 < t_3 \ldots)$, and inscribed in the curve $(a \leqslant t \leqslant b)$, provided that its segment lengths decrease indefinitely, is called the *length of the arc* (segment of the curve).

Any segment of a smooth curve is of certain length. If the curve is given by an equation $\mathbf{r} = \mathbf{r}(t)$, *then the length of a segment* $a \leqslant t \leqslant b$ *of the curve is determined by the formula*

$$s = \int_a^b |\mathbf{r}'(t)|\, dt.$$

Proof. The length of the broken line equals

$$\sum_k |\mathbf{r}(t_k)-\mathbf{r}(t_{k-1})|$$

$$=\int_a^b |\mathbf{r}'(t)|\,dt+\left\{\sum_k (t_k-t_{k-1})\,|\mathbf{r}'(t_k)|-\int_a^b |\mathbf{r}'(t)|\,dt\right\}$$

$$+\left\{\sum_k |\mathbf{r}(t_k)-\mathbf{r}(t_{k-1})|-\sum_k (t_k-t_{k-1})\,|\mathbf{r}'(t_k)|\right\}.$$

The second term on the right-hand side is arbitrarily small for sufficiently small $t_k - t_{k-1}$ by definition of integral, whereas the third term admits a representation in the form

$$\sum_k \left|\int_{t_{k-1}}^{t_k} \mathbf{r}'(t)\,dt\right|-\sum_k \left|\int_{t_{k-1}}^{t_k} \mathbf{r}'(t_k)\,dt\right|;$$

therefore, it does not exceed

$$\sum_k \int_{t_{k-1}}^{t_k} |\mathbf{r}'(t)-\mathbf{r}'(t_k)|\,dt,$$

the difference between the vector moduli being not greater than the modulus of their difference by the "triangle inequality".

Since the vector function $\mathbf{r}'(t)$ is continuous, and, therefore, uniformly continuous on the interval $a \leqslant t \leqslant b$, we obtain $|\mathbf{r}'(t) - \mathbf{r}'(t_k)| < \varepsilon$. Hence, the third term does not exceed

$$\int_a^b \varepsilon\,dt=(b-a)\,\varepsilon.$$

Summing up, we conclude that if the segments of a broken line decrease indefinitely, then the differences $t_k - t_{k-1}$ also decrease, and the length of the broken line tends to the limit

$$s=\int_a^b |\mathbf{r}'(t)|\,dt.$$

Q.E.D.

We now give formulas for the length of a curve in various cases of its specification, viz.,

(i) given by the equations

$$x = x(t), \quad y = y(t), \quad z = z(t),$$

$$s(t_1, t_2) = \int_{t_1}^{t_2} \sqrt{x'^2 + y'^2 + z'^2}\, dt,$$

(ii) given by the equations

$$y = y(x), \quad z = z(x),$$

$$s(x_1, x_2) = \int_{x_1}^{x_2} \sqrt{1 + y'^2 + z'^2}\, dx.$$

For plane curves in the xy-plane, we have to set $z' = 0$.

2. Natural Parametrization of a Curve

Let γ be a smooth curve given by a vector equation

$$\mathbf{r} = \mathbf{r}(t).$$

We introduce a function $s(t)$ in accordance with the formula

$$s(t) = \int_{t_0}^{t} |\mathbf{r}'(t)|\, dt.$$

This function has a simple geometric meaning, viz., $|s(t)|$ is the length of the segment $[t_0, t]$ of γ. $s(t)$ is strictly monotonic, since

$$\frac{ds}{dt} = |\mathbf{r}'(t)| > 0.$$

Therefore, s can be taken as the parameter of the curve. Such a parametrization of the curve is said to be *natural*.

In the natural parametrization case, the tangent vector to the curve $\mathbf{r}'(s)$ is unit, i.e., $|\mathbf{r}'(s)| = 1$.

In fact,

$$\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt} \cdot \frac{dt}{ds} = \frac{\mathbf{r}'}{s'},$$

but $s' = |\mathbf{r}'|$; therefore, $\left|\frac{d\mathbf{r}}{ds}\right| = 1.$

3. Curvature

Let P be an arbitrary point of a regular curve γ, and Q its point near to P. Denote by $\Delta\theta$ the angle between the tangents at P and Q, and by $|\Delta s|$ the length of the arc PQ (Fig. 92).

The limit of the ratio $\Delta\theta / |\Delta s|$ as $Q \to P$ is called the *curvature* of γ at P.

A regular (twice continuously differentiable) curve has certain curvature k_1 *at each point. If*

$$\mathbf{r} = \mathbf{r}(s)$$

is the natural parametrization of the curve, then

$$k_1 = |\mathbf{r}''(s)|.$$

Let two points P and Q be associated with values s and $s + \Delta s$ of the parameters. The angle $\Delta\theta$ equals that between the unit tangent vectors $\boldsymbol{\tau}(s) = \mathbf{r}'(s)$ and $\boldsymbol{\tau}(s + \Delta s) = \mathbf{r}'(s + \Delta s)$.

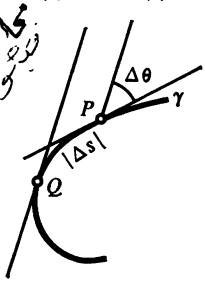

Fig. 92

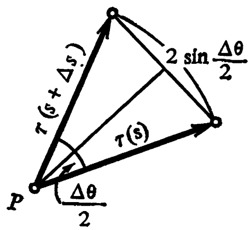

Fig. 93

Since $\boldsymbol{\tau}(s)$ and $\boldsymbol{\tau}(s + \Delta s)$ are unit, and make the angle $\Delta\theta$, $|\boldsymbol{\tau}(s + \Delta s) - \boldsymbol{\tau}(s)| = 2 \sin \frac{\Delta\theta}{2}$ (Fig. 93).

Hence,

$$\frac{|\boldsymbol{\tau}(s+\Delta s) - \boldsymbol{\tau}(s)|}{|\Delta s|} = \frac{2 \sin \frac{\Delta\theta}{2}}{|\Delta s|} = \frac{\sin \frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}} \cdot \frac{\Delta\theta}{|\Delta s|}.$$

Noticing that $\Delta\theta \to 0$ as $|\Delta s| \to 0$ by the continuity of $\boldsymbol{\tau}(s)$ and passing to the limit, we obtain

$$|\mathbf{r}''(s)| = k_1.$$

Q.E.D.

Let the curvature be other than zero at a given point. Consider the vector $\boldsymbol{\nu} = \frac{1}{k_1} \mathbf{r}''(s)$. $\boldsymbol{\nu}$ is unit, and placed in the osculating plane. Besides, it is perpendicular to the tangent vector $\boldsymbol{\tau}$, since $\boldsymbol{\tau}^2 = 1$, and, therefore, $\boldsymbol{\tau}\boldsymbol{\tau}' = \boldsymbol{\tau}\boldsymbol{\nu} k_1 = 0$. Thus, it has the direction of the principal normal, obviously unaltered if the point from which arcs s are counted off or the reference direction are changed. Speaking of the unit principal normal vector in the sequel, we will mean $\boldsymbol{\nu}$.

It is obvious that the vector $\boldsymbol{\tau} \wedge \boldsymbol{\nu} = \boldsymbol{\beta}$ has the same direction as the binormal to the curve. We will call it the *unit binormal vector.*

We now find the expression for curvature with any specification of the curve. Let it be given by a vector equation

$$\mathbf{r} = \mathbf{r}\,(t).$$

Express the second derivative of the vector function $\mathbf{r}$ with respect to the arc s in terms of the derivatives with respect to t.

We have

$$\mathbf{r}' = \mathbf{r}'_s s'.$$

Hence,

$$\mathbf{r}'^2 = s'^2.$$

Therefore,

$$\mathbf{r}'_s = \frac{\mathbf{r}'}{\sqrt{\mathbf{r}'^2}}.$$

Differentiating the equality with respect to t once again, we obtain

$$\mathbf{r}''_{ss} s' = \frac{\mathbf{r}''}{\sqrt{\mathbf{r}'^2}} - \frac{(\mathbf{r}'\mathbf{r}'')\,\mathbf{r}'}{(\sqrt{\mathbf{r}'^2})^3}.$$

Squaring and making note of $s'^2 = \mathbf{r}'^2$, we have

$$k_1^2 = \frac{\mathbf{r}''^2\mathbf{r}'^2 - (\mathbf{r}'\mathbf{r}'')^2}{(\mathbf{r}'^2)^3},$$

or, which is equivalent,

$$k_1^2 = \frac{(\mathbf{r}' \wedge \mathbf{r}'')^2}{(r'^2)^3}.$$

It follows for the curvature of a curve given by equations

$$x = x\,(t), \quad y = y\,(t), \quad z = z\,(t)$$

that

$$k_1^2 = \frac{\begin{vmatrix} x'' & y'' \\ x' & y' \end{vmatrix}^2 + \begin{vmatrix} y'' & z'' \\ y' & z' \end{vmatrix}^2 + \begin{vmatrix} z'' & x'' \\ z' & x' \end{vmatrix}^2}{(x'^2 + y'^2 + z'^2)^3}$$

If the curve is plane and placed in the xy-plane, then

$$k_1^2 = \frac{(x''y' - y''x')^2}{(x'^2 + y'^2)^3},$$

and if a plane curve is given by an equation of the form $y = y\,(x)$, then

$$k_1^2 = \frac{y''^2}{(1 + y'^2)^3}$$

Remark. By definition, curvature is non-negative, it is useful to assume, however, that it may be positive for some plane curves and negative for others. Meanwhile, we will adopt the following argument. In moving along the curve, the tangent vector $\mathbf{r}'(t)$ turns in the direction of increasing t. Depending on the sense of rotation, curvature is regarded either as positive or negative (Fig. 94). If the curvature sign for a plane curve is determined just by this condition, then we obtain either

$$k=\frac{x''y'-y''x'}{(x'^2+y'^2)^{3/2}} \text{ or } k=-\frac{x''y'-y''x'}{(x'^2+y'^2)^{3/2}}.$$

In particular,

$$k=\frac{y''}{(1+y'^2)^{3/2}} \text{ or } k=-\frac{y''}{(1+y'^2)^{3/2}}$$

if the curve is given by an equation of the form $y = y(x)$.

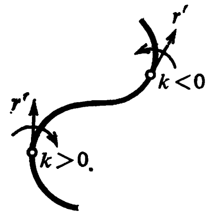

Fig. 94

As an exercise, we find all the curves of zero curvature at each point.

We have

$$k_1 = |\mathbf{r}''(s)| = 0.$$

Hence, $\mathbf{r}''(s) = 0$, and $\mathbf{r}(s) = \mathbf{a}s + \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors.

Thus, a curve with zero curvature everywhere is either a straight line or a straight line segment.

4. Torsion of a Curve

Let P be an arbitrary point of a curve γ, and Q its point near to P. Denote the angle between the osculating planes at P and Q by $\Delta\theta$, and the length of the curve segment PQ by $|\Delta s|$. By the *absolute value of torsion* $|k_2|$ of γ at P, we understand the limit of the ratio $\Delta\theta/|\Delta s|$ as $Q \to P$ (Fig. 95).

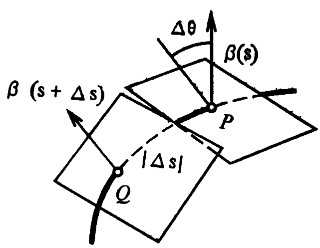

Fig. 95

A regular (thrice continuously differentiable) curve has certain absolute torsion $|k_2|$ *at each point where the curvature is other than zero. If*

$$\mathbf{r} = \mathbf{r}(s)$$

is the natural parametrization, then

$$|k_2| = \frac{|(\mathbf{r}'\mathbf{r}''\mathbf{r}''')|}{k_1^2}.$$

Proof. If the curvature of the curve γ at P is different from zero, then, by continuity, it is also other than zero at points near to P. The vectors $\mathbf{r}'(s)$ and $\mathbf{r}''(s)$ are non-zero and non-parallel at each point with non-zero curvature. Therefore, there exists a certain osculating plane at each point Q close to P.

Let $\beta(s)$ and $\beta(s+\Delta s)$ be the unit binormal vectors at two points P and Q of γ. $\Delta\theta$ equals the angle between $\beta(s)$ and $\beta(s+\Delta s)$.

Since $\beta(s)$ and $\beta(s+\Delta s)$ are unit, make the angle $\Delta\theta$, we have $|\beta(s+\Delta s)-\beta(s)| = 2\sin\frac{\Delta\theta}{2}$.

Therefore,

$$\frac{|\beta(s+\Delta s)-\beta(s)|}{|\Delta s|} = \frac{2\sin\frac{\Delta\theta}{2}}{|\Delta s|} = \frac{\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}}\cdot\frac{\Delta\theta}{|\Delta s|}.$$

Hence, passing to the limit as $|\Delta s|\to 0$, we obtain

$$|k_2| = |\beta'|.$$

The vector β' is perpendicular to β, since $\beta'\beta = \left(\frac{1}{2}\beta^2\right) = 0$. It is easy to see that it is also perpendicular to τ.

Indeed,

$$\beta' = (\tau\wedge\nu)' = \tau'\wedge\nu+\tau\wedge\nu'.$$

However, $\tau' \parallel \nu$. Therefore, $\beta' = \tau\wedge\nu'$, in which case β' is perpendicular to τ. Thus β' is parallel to ν; hence,

$$|k_2| = |\beta'\nu|.$$

Substituting $\nu = \frac{1}{k_1}\mathbf{r}''$ and $\beta = \frac{\mathbf{r}'\wedge\mathbf{r}''}{k_1}$, we derive

$$|k_2| = \frac{|(\mathbf{r}'\mathbf{r}''\mathbf{r}''')|}{k_1^2}.$$

Q.E.D.

We now define the *torsion* of a curve.

It follows from the parallelism of β' and ν that, in moving along a curve in the direction of increasing s, the osculating plane of the curve rotates about the tangent, due to which we define the torsion of the curve by the equality

$$k_2 = \pm|k_2|,$$

take the plus if the rotation of the tangent plane is in the direction from β to ν (Fig. 96), and the minus if it occurs in the direction from ν to β. With this definition of the torsion of a curve, we will have either $k_2 = \beta'\nu$ or $k_2 = -\frac{(\mathbf{r}'\mathbf{r}''\mathbf{r}''')}{k_1^2}$.

We now find the expression for the torsion of a curve in the case of any regular parametrization $\mathbf{r} = \mathbf{r}(t)$.

We have

$$\mathbf{r}'_s = \mathbf{r}'t', \quad \mathbf{r}''_{ss} = \mathbf{r}''t'^2+\mathbf{r}'t'', \quad \mathbf{r}'''_{sss} = \mathbf{r}'''t'^3+\{\mathbf{r}', \mathbf{r}''\},$$

where $\{\mathbf{r}', \mathbf{r}''\}$ is a linear combination of the vectors $\mathbf{r}'$ and $\mathbf{r}''$. Substituting the expressions obtained for $\mathbf{r}'_s$, $\mathbf{r}''_{ss}$ and $\mathbf{r}'''_{sss}$ in the formula for k_2, and noticing that $t'^2 = 1/(\mathbf{r}'^2)$, we obtain

$$k_2 = -\frac{(\mathbf{r}'\mathbf{r}''\mathbf{r}''')}{(\mathbf{r}' \wedge \mathbf{r}'')^2}.$$

We now find all the curves with zero torsion at each point.

We have

$$k_2 = \boldsymbol{\beta}'\boldsymbol{\nu} = 0.$$

Besides, since $\boldsymbol{\beta}'\boldsymbol{\tau} = 0$, and $\boldsymbol{\beta}'\boldsymbol{\beta} = 0$, we obtain $\boldsymbol{\beta}' = 0$, $\boldsymbol{\beta} = \boldsymbol{\beta}_0 = \text{const}$.

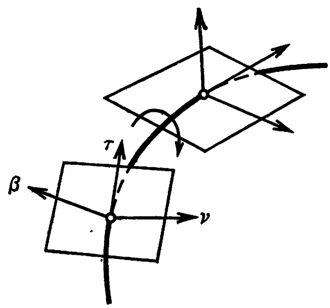

Fig. 96

The vectors $\boldsymbol{\tau}$ and $\boldsymbol{\beta}$ are perpendicular. Hence, $\mathbf{r}'\boldsymbol{\beta}_0 = 0$ and $(\mathbf{r}(s) - \mathbf{r}_0)\boldsymbol{\beta}_0 = 0$, which means that the curve is in the plane given by the vector equation $(\mathbf{r} - \mathbf{r}_0)\boldsymbol{\beta}_0 = 0$.

Thus, *a curve with zero torsion at each point is plane.*

5. Frenet Formulas

Three half-lines emanating from a point on the curve, and with the directions of three vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$, are the edges of a trihedral angle called a *moving trihedral.*

Express the derivatives of $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$ with respect to the arc of the curve in terms of $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$ themselves.

We have

$$\boldsymbol{\tau}' = \mathbf{r}'' = k_1\boldsymbol{\nu}.$$

To obtain $\boldsymbol{\beta}'$, we recall that it is parallel to $\boldsymbol{\nu}$, and $\boldsymbol{\beta}'\boldsymbol{\nu} = k_2$. Hence,

$$\boldsymbol{\beta}' = k_2\boldsymbol{\nu}.$$

Finally,

$$\begin{aligned}\boldsymbol{\nu}' &= (\boldsymbol{\beta} \wedge \boldsymbol{\tau})' = \boldsymbol{\beta}' \wedge \boldsymbol{\tau} + \boldsymbol{\beta} \wedge \boldsymbol{\tau}' \\ &= k_2\boldsymbol{\nu} \wedge \boldsymbol{\tau} + k_1\boldsymbol{\beta} \wedge \boldsymbol{\nu} = -(k_1\boldsymbol{\tau} + k_2\boldsymbol{\beta}).\end{aligned}$$

The formulas

$$\begin{aligned}\boldsymbol{\tau}' &= \quad k_1\boldsymbol{\nu}, \\ \boldsymbol{\nu}' &= -k_1\boldsymbol{\tau} - k_2\boldsymbol{\beta}, \\ \boldsymbol{\beta}' &= \quad k_2\boldsymbol{\nu}\end{aligned}$$

are called the *Frenet formulas.*

The curvature and torsion of a curve are functions of arc length s along the curve. The equations

$$k_1 = \varphi(s), \quad k_2 = \psi(s)$$

specifying the curvature and torsion of the curve as functions of arc length s are called the *natural equations of the curve.*

It turns out that *a curve is determined uniquely up to position in space by its natural equations* if $k_1 > 0$.

6. Evolute and Evolvent of a Plane Curve

Let γ be a regular (thrice differentiable) curve given by an equation of the form $\mathbf{r} = \mathbf{r}(s)$. Cut off a line segment equal to the curvature radius $\rho = \frac{1}{k_1}$ in the direction of the vector $\mathbf{v}$, on the normal to the curve from its arbitrary point P. We call the segment's end-point the *centre of curvature* of the curve. The name is due to contact of order three between the circle with this centre and radius ρ, and the curve at P, i.e., the distance from a point on the curve to the circle is an infinitesimal of order three with respect to the distance from P. Recall that a tangent has contact of order two with the curve.

If it is a curve, then the locus of the centres of curvature is called an *evolute.* We show that an evolute is the envelope of the normals to a curve. In fact, the equation of an evolute is

$$\tilde{\mathbf{r}} = \mathbf{r} + \frac{1}{k_1}\mathbf{v},$$

whereas the tangent vector of the evolute is

$$\tilde{\mathbf{r}}' = \mathbf{r}' + \left(\frac{1}{k_1}\right)'\mathbf{v} + \frac{1}{k_1}(-k_1\boldsymbol{\tau}) = \left(\frac{1}{k_1}\right)'\mathbf{v},$$

and is thus directed along a normal to the curve. Therefore, the normal is a tangent to the evolute, which means that the *evolute of a curve is the envelope of its normals.*

Note that if $k_1' \neq 0$, then the length of the evolute $a \leqslant s \leqslant b$ equals

$$\int_a^b |\tilde{\mathbf{r}}'|\,ds = \int_a^b \left|\left(\frac{1}{k_1}\right)'\right| ds = \left|\frac{1}{k_1(b)} - \frac{1}{k_1(a)}\right|,$$

or the difference between the curvature radii at the ends of the segment.

We now define the evolvent of a curve. Let $\mathbf{r} = \mathbf{r}(s)$ be a curve with the natural parametrization. If $s < 0$, then we mark off a line segment of length $|s|$ from a point on the curve on its tangent along the direction of the vector $\boldsymbol{\tau}$, and along the opposite direction if $s > 0$. The curve described by the end of this segment is called the *evolvent* of the curve.

The formation of an evolvent can be visualized as follows. Imagine an inextensible string fixed on a curve with one end, and wound around it. If we wind the thread off the curve by pulling a tits free end, then it describes the evolvent of the curve (Fig. 97).

This curve is the evolute for its evolvent. Indeed, the equation of an evolvent is

$$\tilde{\mathbf{r}} = \mathbf{r} - s\boldsymbol{\tau},$$

whereas the tangent vector of the evolvent is

$$\tilde{\mathbf{r}}' = \mathbf{r} - \boldsymbol{\tau} - sk_1\boldsymbol{\nu} = -sk_1\boldsymbol{\nu}.$$

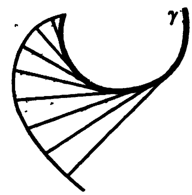

Fig. 97

It follows that a tangent to a curve is a normal to the evolvent. Therefore, the given curve is the evolute for its evolvent.

Evolutes and evolvents have found important areas of practical applications, e.g., the teeth of cylindrical gear wheels have the form of the evolvents of circles.

EXERCISES TO CHAPTER IX

1. Find the length of a segment $-a \leqslant x \leqslant a$ of a parabola $y = bx^2$.

2. Find the length of the segment of a curve

$$x = a \cosh t, \quad y = a \sinh t, \quad z = at$$

between the points 0 and t.

3. Find the length of the astroid

$$x = a \cos^3 t, \quad y = a \sin^3 t.$$

4. Find the length of the segment $0 \leqslant t \leqslant 2\pi$ of the cycloid

$$x = a(t - \sin t), \quad y = a(1 - \cos t).$$

5. Find the expression for the arc length of a curve given in polar coordinates by an equation $\rho = \varphi(\theta)$.

6. Find the curvature of the curve

$$x = t - \sin t, \quad y = 1 - \cos t, \quad z = 4 \sin \frac{t}{2}.$$

7. Find the curvature of the curve given by the implicit equations

$$x + \sinh x = \sin y + y, \quad z + e^z = x + \ln(1 + x) + 1$$

at the point (0, 0, 0).

8. Find the curvature of a circle of radius R.

9. Find the curvature and torsion of

$$x = a \cosh t, \quad y = a \sinh t, \quad z = at.$$

10. Find the curvature of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

at its vertices.

11. Show that the curvature and torsion of a helix are constant.

12. Derive a formula for the curvature of the plane curve given in polar coordinates by the equation

$$k_1=\frac{\left|\frac{1}{\rho}+\left(\frac{1}{\rho}\right)''\right|}{\left(1+\left(\frac{\rho'}{\rho}\right)^2\right)^{3/2}}.$$

13. Prove that if the tangents to a curve make a constant angle with a certain straight line, then the principal normals are perpendicular to the line.

14. Prove that if the osculating planes of a curve are concurrent, then the torsion of the curve is zero, and, therefore, the curve is plane.

15. Find the torsion of the curve

$$\mathbf{r}=\int \mathbf{e}(t)\wedge\mathbf{e}'(t)\,dt,$$

where $\mathbf{e}(t)$ is a vector function satisfying $|\mathbf{e}(t)|=1$, $\mathbf{e}'(t)\neq 0$.

16. Prove that if the tangents to a curve make a constant angle with a certain direction, then the ratio of the curvature to the torsion is constant.

17. Find the evolute of the parabola

$$y^2=2px.$$

18. Show that the evolute of the tractrix

$$x=-a\left(\ln\tan\frac{t}{2}+\cos t\right),\quad y=a\sin t$$

is the *catenary curve* $y=a\cosh\frac{x}{a}$.

19. Find the evolvents of the circle $x^2+y^2=R^2$.

20. Find all the plane curves with the given natural equation $k_1=k(s)$.

21. How can the equations of a curve be found, given one of the three vector functions $\boldsymbol{\tau}(s)$, $\mathbf{v}(s)$ or $\boldsymbol{\beta}(s)$?

22. Prove that if a curve possesses one of the following four properties, viz.,

(a) the tangents make a constant angle with a certain direction,

(b) the binormals make a constant angle with a certain direction,

(c) the principal normals are parallel to a certain plane,

and

(d) the ratio of curvature to torsion is constant, then it possesses the other three properties.

23. Prove that if the curvature and torsion of a curve are constant, then this is a helix.

Chapter X

TANGENT PLANE AND OSCULATING PARABOLOID OF SURFACE

1. Concept of Surface

Let G be a set of points in the plane. A point X of G is said to be *interior* if all points of the plane, which are sufficiently near to X, belong to G. This means that there is a positive number ε such that all points in the plane, whose distance from X is less than ε, lie in G. A set G is said to be *open* if each of its points is interior. A set G is called a *domain* if it is open, and if any two of its points can be joined with a broken line lying in G. E.g., a circle without its boundary circumference is a domain.

Let G be a domain in the plane. A point X of the plane is said to be *boundary* for G if there are points in G, which are arbitrarily near to X, and if there are points not belonging to G, which means that, for any $\varepsilon > 0$, there are points belonging to G, which are from X at a distance less than ε, and if there are points not in G. The boundary points make up the *boundary* of the domain G. In the above example, the circumference bounding a circle consists of boundary points. Annexing the boundary to a domain, we obtain a *closed* domain.

The concepts of interior point of a set in space, of open set, domain and closed domain are defined verbatim as for planar sets. A *neighbourhood* of a point is any open set containing the point. In particular, an ε-neighbourhood is the set of points which are from a given point at a distance less than ε.

We now give a few definitions related to the concept of surface.

We will call a figure obtained by a topological transformation of a plane domain an *elementary surface*. A figure is called a *simple surface* if each of its points possesses a three-dimensional neighbourhood such that part of the figure, contained in the neighbourhood, is an elementary surface. A *generic surface* is a figure obtained by a locally topological transformation of a simple surface.

Due to these definitions, the study of any surface "in the small" reduced to that of an elementary surface.

Let an elementary surface F be obtained by a topological transformation of a plane domain G. Introduce Cartesian coordinates u, v in the plane of G. The equations

$$x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v) \qquad (*)$$

specifying a transformation of G into F are said to be *parametric*.

The values u and v completely specify the position of a point in the surface, and are called *curvilinear*, or *Gauss*, coordinates on the surface.

For fixed u (or v), the equations (*) specify certain curves in the surface. They are called *coordinate lines*. The lines along which only u varies (v = const) are called the *u-curves*, whereas those along which only v varies (u = const) are termed the *v-curves*.

Specifying a surface by the equations

$$x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v)$$

is equivalent to that by one vector equation

$$\mathbf{r} = \mathbf{f}(u, v),$$

where

$$\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3,$$

$$\mathbf{f}(u, v) = f_1(u, v)\,\mathbf{e}_1 + f_2(u, v)\,\mathbf{e}_2 + f_3(u, v)\,\mathbf{e}_3.$$

2. Regular Surfaces

We call a surface F *regular* if each of its points possesses a neighbourhood admitting a regular parametrization, or parametric representation

$$x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v),$$

where f_1, f_2, f_3 are regular functions (k times continuously differentiable, $k \geqslant 1$) such that the rank of the matrix

$$\begin{pmatrix} f_{1u} & f_{2u} & f_{3u} \\ f_{1v} & f_{2v} & f_{3v} \end{pmatrix}$$

is two, i.e., at least one of the determinants of order two is not zero. In the case of vector specification by an equation $\mathbf{r} = \mathbf{f}(u, v)$, this means that $\mathbf{f}_u \wedge \mathbf{f}_v \neq \mathbf{0}$, and that the vectors $\mathbf{f}_u$ and $\mathbf{f}_v$ are non-zero and non-collinear. When $k = 1$, the surface is said to be *smooth*.

Let a smooth surface be given by parametric equations

$$x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v),$$

and

$$\begin{vmatrix} f_{1u} & f_{2u} \\ f_{1v} & f_{2v} \end{vmatrix} \neq 0$$

at a point $Q_0(u_0, v_0)$.

We prove that *the surface in the neighbourhood of* Q_0 *admits a specification by an equation of the form*

$$z = F(x, y).$$

By the implicit function theorem, the system of equations

$$x = f_1(u, v), \quad y = f_2(u, v)$$

can be solved for u, v in the neighbourhood of (u_0, v_0). We obtain

$$u = \varphi(x, y), \quad v = \psi(x, y).$$

Introducing parameters α, β instead of u, v according to the formulas $u = \varphi(\alpha, \beta)$, $v = \psi(\alpha, \beta)$, we get

$$x = \alpha, \quad y = \beta, \quad z = f_3(\varphi(\alpha, \beta), \quad \psi(\alpha, \beta)),$$

or, which is equivalent,

$$z = f_3(\varphi(x, y), \quad \psi(x, y)) = F(x, y),$$

and the statement is thus proved.

3. Tangent Plane to a Surface

Let Φ be a surface, P a point in it, and α a plane passing through P (Fig. 98). Take another point Q in the surface, and denote its distances from P and α by d and h, respectively.

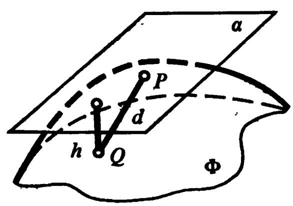

Fig. 98

We will call α the *tangent plane* to the surface at P if the ratio $h/d \to 0$ as $Q \to P$.

A smooth surface Φ *possesses one, and only one, tangent plane at each point.*

If $\mathbf{r} = \mathbf{r}(u, v)$ *is some smooth parametrization of the surface, then the tangent plane at a point* $P(u, v)$ *is parallel to the vectors* $\mathbf{r}_u(u, v)$ and $\mathbf{r}_v(u, v)$.

Proof. Assume that Φ at $P(u, v)$ possesses a tangent plane α. Let $\mathbf{n}$ be the unit vector perpendicular to α. The distance d from a point $Q(u + \Delta u, v + \Delta v)$ to $P(u, v)$ is equal to $|\mathbf{r}(u + \Delta u, v + \Delta v) - \mathbf{r}(u, v)|$, whereas that from Q to α is

$$|(\mathbf{r}(u+\Delta u, v+\Delta v) - \mathbf{r}(u, v))\mathbf{n}|,$$

$$\frac{h}{d} = \frac{|(\mathbf{r}(u+\Delta u, v+\Delta v) - \mathbf{r}(u, v))\mathbf{n}|}{|\mathbf{r}(u+\Delta u, v+\Delta v) - \mathbf{r}(u, v)|}.$$

By definition, $h/d \to 0$ as Δu and Δv independently tend to zero. In particular,

$$\frac{|[\mathbf{r}(u+\Delta u, v) - \mathbf{r}(u, v))\mathbf{n}|}{|\mathbf{r}(u+\Delta u, v) - \mathbf{r}(u, v)|} \to 0 \text{ as } \Delta u \to 0.$$

However,

$$\frac{|(\mathbf{r}(u+\Delta u, v)-\mathbf{r}(u, v))\,\mathbf{n}|}{|\mathbf{r}(u+\Delta u, v)-\mathbf{r}(u, v)|}=\frac{\left|\dfrac{\mathbf{r}(u+\Delta u, v)-\mathbf{r}(u, v)}{\Delta u}\cdot\mathbf{n}\right|}{\left|\dfrac{\mathbf{r}(u+\Delta u, v)-\mathbf{r}(u, v)}{\Delta u}\right|}\to\frac{|\mathbf{r}_u(u, v)\,\mathbf{n}|}{|\mathbf{r}_u(u, v)|}.$$

Thus,

$$\mathbf{r}_u(u, v)\,\mathbf{n}=0.$$

Since $\mathbf{r}_u(u, v)\neq\mathbf{0}$ $(\mathbf{r}_u\wedge\mathbf{r}_v\neq\mathbf{0})$, the equality $\mathbf{r}_u(u, v)\,\mathbf{n}=0$ holds if and only if $\mathbf{r}_u(u, v)$ is parallel to α. It is shown similarly that $\mathbf{r}_v(u, v)$ is also parallel to α, and, since both are non-zero and non-parallel, i.e., $(\mathbf{r}_u\wedge\mathbf{r}_v)\neq\mathbf{0}$, the tangent plane is unique if it exists.

We now prove the existence of a tangent plane. Let a plane α be parallel to $\mathbf{r}_u(u, v)$ and $\mathbf{r}_v(u, v)$. We show that it is tangent to the surface at the point $P(u, v)$.

We have

$$\frac{h}{d}=\frac{|(\mathbf{r}(u+\Delta u, v+\Delta v)-\mathbf{r}(u, v))\,\mathbf{n}|}{|\mathbf{r}(u+\Delta u, v+\Delta v)-\mathbf{r}(u, v)|}$$

$$=\frac{|(\mathbf{r}_u\mathbf{n})\,\Delta u+(\mathbf{r}_v\mathbf{n})\,\Delta v+\varepsilon_1\sqrt{\Delta u^2+\Delta v^2}|}{|\mathbf{r}_u\,\Delta u+\mathbf{r}_v\,\Delta v+\varepsilon_2\sqrt{\Delta u^2+\Delta v^2}|}$$

$$=\frac{|\varepsilon_1|}{\left|\mathbf{r}_u\dfrac{\Delta u}{\sqrt{\Delta u^2+\Delta v^2}}+\mathbf{r}_v\dfrac{\Delta v}{\sqrt{\Delta u^2+\Delta v^2}}+\varepsilon_2\right|},$$

where $|\varepsilon_1|$ and $|\varepsilon_2|$ tend to zero as $\Delta u, \Delta v\to 0$.

To prove that $h/d\to 0$ as $\Delta u, \Delta v\to\mathbf{0}$, it suffices to show that, for any Δu and Δv,

$$\left|\frac{r_u\,\Delta u}{\sqrt{\Delta u^2+\Delta v^2}}+\frac{r_v\,\Delta v}{\sqrt{\Delta u^2+\Delta v^2}}\right|>c>0,$$

where c is a certain constant.

Since the sum of the squares of $\Delta u/\sqrt{\Delta u^2+\Delta v^2}$ and $\Delta v/\sqrt{\Delta u^2+\Delta v^2}$ is unity, at least one of them is not less than $1/\sqrt{2}$. E.g., let $\Delta u/\sqrt{\Delta u^2+\Delta v^2}\geqslant 1/\sqrt{2}$. Denote by $\mathbf{e}$ the unit vector coplanar with the vectors $\mathbf{r}_u$ and $\mathbf{r}_v$, and perpendicular to $\mathbf{r}_v$. We have

$$\left|\frac{\mathbf{r}_u\,\Delta u}{\sqrt{\Delta u^2+\Delta v^2}}+\frac{\mathbf{r}_v\,\Delta v}{\sqrt{\Delta u^2+\Delta v^2}}\right|$$

$$\geqslant\left|\left(\mathbf{r}_u\frac{\Delta u}{\sqrt{\Delta u^2+\Delta v^2}}+\mathbf{r}_v\frac{\Delta v}{\sqrt{\Delta u^2+\Delta v^2}}\right)\mathbf{e}\right|$$

$$=\left|(\mathbf{r}_u\mathbf{e})\frac{\Delta u}{\sqrt{\Delta u^2+\Delta v^2}}\right|\geqslant|\mathbf{r}_u|\frac{\sin\theta}{\sqrt{2}},$$

where θ is the angle between $\mathbf{r}_u$ and $\mathbf{r}_v$. Similarly, if $\Delta v/\sqrt{\Delta u^2+\Delta v^2}\geqslant 1/\sqrt{2}$, then

$$\left|\frac{\mathbf{r}_u\,\Delta u}{\sqrt{\Delta u^2+\Delta v^2}}+\frac{\mathbf{r}_v\,\Delta v}{\sqrt{\Delta u^2+\Delta v^2}}\right|\geqslant|\mathbf{r}_v|\,\frac{\sin\theta}{\sqrt{2}}.$$

Thus, we can take the least of the values $|\mathbf{r}_u|\sin\theta/\sqrt{2}$ and $|\mathbf{r}_v|\sin\theta/\sqrt{2}$ as the constant c.

Q.E.D.

4. Equation of a Tangent Plane

We now make up the equation of a tangent plane to a surface given by parametric equations

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v).$$

Let $Q_0(u_0, v_0)$ be a point on the surface, and $A(x, y, z)$ an arbitrary point in the tangent plane at Q_0. Then the vectors $\overrightarrow{Q_0A}$, $\mathbf{r}_u$ and $\mathbf{r}_v$ are coplanar. Therefore, their scalar triple product is zero. Hence, the equation of the tangent plane is

$$\begin{vmatrix} x-x(u_0, v_0) & y-y(u_0, v_0) & z-z(u_0, v_0) \\ x_u(u_0, v_0) & y_u(u_0, v_0) & z_u(u_0, v_0) \\ x_v(u_0, v_0) & y_v(u_0, v_0) & z_v(u_0, v_0) \end{vmatrix} = 0.$$

To derive the equation of a tangent plane to a surface given by an equation $z = f(x, y)$, it suffices to notice that this is only a brief form of specifying the surface parametrically, as

$$x = u, \quad y = v, \quad z = f(u, v).$$

Consequently, the tangent plane equation is

$$\begin{vmatrix} x-x_0 & y-y_0 & z-f(x_0, y_0) \\ 1 & 0 & f_x(x_0, y_0) \\ 0 & 1 & f_y(x_0, y_0) \end{vmatrix} = 0,$$

or

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

We now find it for a surface given implicitly by the equation

$$\varphi(x, y, z) = 0, \quad \varphi_x^2 + \varphi_y^2 + \varphi_z^2 \neq 0.$$

Let

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$$

be some parametric representation of the surface. Differentiating the identity

$$\varphi(x(u, v), \quad y(u, v), \quad z(u, v)) = 0$$

with respect to u and v, we obtain

$$\varphi_x x_u + \varphi_y y_u + \varphi_z z_u = 0$$
$$\varphi_x x_v + \varphi_y y_v + \varphi_z z_v = 0.$$

Hence, the vector with components φ_x, φ_y, φ_z is perpendicular to the vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ and, therefore, to the tangent plane. Knowing the vector perpendicular to the plane, we easily obtain its equation, viz.,

$$(x - x_0)\,\varphi_x(x_0, y_0, z_0) + (y - y_0)\,\varphi_y(x_0, y_0, z_0) + (z - z_0)\,\varphi_z(x_0, y_0, z_0) = 0.$$

A straight line passing through a point P on a surface at right angles with the tangent plane at this point is called the *normal* to the surface at P. Obviously, the normal to a surface has the same direction as the vector $\mathbf{r}_u \wedge \mathbf{r}_v$. Hence, its equation is not difficult to make up.

5. Osculating Paraboloid of a Surface

Let Φ be a regular surface, P a point on it, and U a paraboloid with vertex P and axis coinciding with the normal to the surface at P. Take on Φ a point Q near to P. A straight line passing through Q, and parallel to the axis, intersects the paraboloid at a certain point Q'. Denote by h the distance between Q and Q', and by d that between Q and P. U is said to be *osculating* at a point P of the surface if $h/d^2 \to 0$ as $Q \to P$ (Fig. 99).

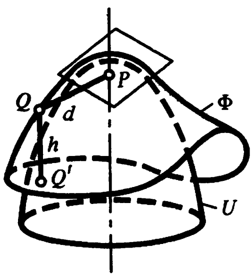

Fig. 99

At each point of regular surface (twice continuously differentiable), there is one, and only one, osculating paraboloid; in particular, it may generate into a parabolic cylinder or plane.

Proof. Let the surface be given by an equation in vector form $\mathbf{r} = \mathbf{r}(u, v)$ (assuming as always that $\mathbf{r}_u \wedge \mathbf{r}_v \neq 0$). We introduce a coordinate system x, y, z by taking as the xy-plane the tangent plane to the surface at a point P, and the normal to the latter as the z-axis. Meanwhile, as the vector $\mathbf{r}_u \wedge \mathbf{r}_v$ is directed along the z-axis,

$$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \neq 0.$$

Therefore, in a sufficiently small neighbourhood of P the surface can be given by an equation of the form

$$z = f(x, y).$$

Since the tangent plane equation at P is

$$z = xf_x(0, 0) + yf_y(0, 0),$$

and this is the xy-plane, $f_x(0, 0) = 0$, $f_y(0, 0) = 0$. Accordingly, the expansion of the function $f(x, y)$ in the neighbourhood of the origin is of the form

$$f(x, y) = \frac{1}{2}(rx^2 + 2sxy + ty^2) + \varepsilon(x, y)(x^2 + y^2),$$

where r, s, t are the second derivatives of $f(x, y)$ at the origin ($r = f_{xx}$, $s = f_{xy}$, $t = f_{yy}$), and $\varepsilon(x, y) \to 0$ as $x, y \to 0$. Thus, the equation of the surface in the neighbourhood of the origin is of the form

$$z = \frac{1}{2}(rx^2 + 2sxy + ty^2) + \varepsilon(x, y)(x^2 + y^2).$$

Any paraboloid with vertex at the origin and the z-axis and also its degeneracy into a parabolic cylinder or plane, can be given by an equation of the form

$$z = ax^2 + 2bxy + cy^2. \qquad (*)$$

We prove that if an osculating paraboloid does exist, then it is unique. Let the paraboloid (*) be osculating.

We have

$$\frac{h}{d^2} = \frac{\left|\frac{1}{2}((r-a)x^2 + 2(s-b)xy + (t-c)y^2) + \varepsilon(x, y)(x^2+y^2)\right|}{x^2 + y^2 + f^2(x, y)}.$$

Putting $y = 0$, and letting $x \to 0$, we see that

$$\frac{h}{d^2} \to \left|\frac{1}{2}(r-a)\right|.$$

Hence, $a = r$. Similarly, we conclude that $c = t$. Setting $x = y \to 0$, we then find that $b = s$. Thus, if an osculating paraboloid exists, then it must have the equation

$$r = \frac{1}{2}(rx^2 + 2sxy + ty^2),$$

and is, therefore, unique.

That it is osculating can be seen easily. Viz.,

$$\frac{h}{d^2} = \frac{|\varepsilon(x, y)(x^2 + y^2)|}{x^2 + y^2 + f^2(x, y)} < |\varepsilon(x, y)| \to 0,$$

thus completing the proof.

6. Classification of Surface Points

The form of a regular surface in a sufficiently small neighbourhood of an arbitrary point is given, to a first approximation, by a tangent plane, and, to a second approximation, by osculating paraboloid. Depending on the osculating paraboloid, the points of a surface are classified into *elliptic*, *hyperbolic*, *parabolic* and *planar* ones.

A point of a surface is said to be *elliptic* if the osculating paraboloid is elliptic at it. In a sufficiently small neighbourhood of such a point, the surface resembles an elliptic paraboloid (Fig. 100*a*).

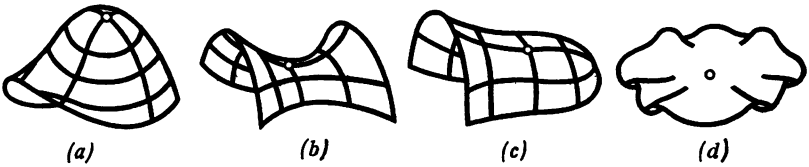

Fig. 100

A point of a surface is said to be *hyperbolic* if the osculating paraboloid at it is hyperbolic (Fig. 100*b*).

A point of a surface is said to be *parabolic* if the osculating paraboloid degenerates into a *parabolic cylinder* (Fig. 100*c*).

A point of a surface is said to be *planar* if the osculating paraboloid degenerates into a plane (which is tangent to the surface) (Fig. 100*d*).

Let P be an elliptic point of a surface. Construct an osculating paraboloid at P and cut the surface with a plane parallel to the tangent one at the point at the distance 1/2 from it, obtaining an ellipse in the section. Its projection onto the tangent plane is called the *Dupin indicatrix*, or the *indicatrix of the normal curvature.*

Since the equation of a paraboloid is $z = \frac{1}{2}(rx^2 + 2sxy + ty^2)$, that of the Dupin indicatrix

$$rx^2 + 2sxy + ty^2 = \pm 1,$$

where the plus and minus depend on where the surface is placed, viz., in the half-space $z > 0$ or $z < 0$.

Directions at a point P on a surface are said to be *conjugate* if they are those of the Dupin indicatrix conjugate diameters at P. Directions at P are said to be *principal* if they are those of the Dupin indicatrix axes at the point.

The Dupin indicatrix of a surface at a hyperbolic point is defined similarly, consisting of two conjugate hyperbolas given by the equation

$$rx^2 + 2sxy + ty^2 = \pm 1,$$

the plus corresponding to one, and the minus to the other (conjugate) hyperbola. In addition to conjugate and principal directions, at a hyperbolic point, we introduce the concept of *asymptotic* directions, viz., those of the indicatrix asymptotes.

At a parabolic point P of the surface, the Dupin indicatrix consists of two parallel straight lines symmetric about P. At a planar point, a Dupin indicatrix does not exist.

The name "Dupin indicatrix" is related to the French geometer Ch. Dupin who introduced the concept. The term "indicatrix of the normal curvature" will be made clear by what follows.

EXERCISES TO CHAPTER X

1. Given the circle

$$z^2 + (x - a)^2 = R^2, \quad a > R,$$

in the xz plane, find the equation of the surface obtained by rotating it about the z-axis (*torus*).

2. Determine the form of the surface given parametrically by

$$x = a \cos u \cos v, \quad y = a \cos u \sin v, \quad z = c \sin u,$$

and find its implicit equation.

3. Find the equation of the surface obtained by rotating a curve $x = \varphi(u)$, $z = \psi(u)$ about the z-axis (*surface of revolution*).

4. A straight line g moves in space so that

(a) it always intersects the z-axis at right angles,

(b) the point where g meets the z-axis moves uniformly with velocity a,

(c) g rotates uniformly about the z-axis with angular velocity ω.

Find the equation of the surface described by g (*helical surface, helicoid*).

5. What is the form of the surface formed by the principal normals to a helix?

6. The surface formed by translating a curve along another is called a *translation surface*. Prove that a translation surface can be given by an equation of the form $\mathbf{r} = \boldsymbol{\varphi}(u) + \boldsymbol{\psi}(v)$, where $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are two vector functions, of which $\boldsymbol{\varphi}$ depends only on u, and $\boldsymbol{\psi}$ only on v.

7. Show that the locus of the midpoints of line segments whose ends are on two given curves is a translation surface.

8. Make up the equation of the surface formed by straight lines parallel to a vector $\mathbf{a}$, and intersecting a curve $\mathbf{r} = \mathbf{r}(u)$ (*cylindrical surface*).

9. Find the equation of the surface formed by straight lines passing through a point (a, b, c), and intersecting a curve $\mathbf{r} = \mathbf{r}(u)$ (*conical surface*).

10. Show that the equation of any surface formed by straight lines can be written as $\mathbf{r} = \mathbf{f}(u) + v\boldsymbol{\varphi}(u)$, where $\mathbf{f}$ and $\boldsymbol{\varphi}$ are two vector functions.

11. Show that the equation of the tangent plane at a point (x_0, y_0, z_0) on a surface $ax^2 + by^2 + cz^2 = 1$ can be written as $axx_0 + byy_0 + czz_0 = 1$.

12. Make up the equation of the tangent plane to a sphere

$$x = a \cos u \cos v, \quad y = a \cos u \sin v, \quad z = a \sin u$$

at the point $(a, 0, 0)$.

13. Show that all the tangent planes to the surface $z = x\varphi\left(\frac{y}{x}\right)$ pass through the origin of coordinates.

14. Show that the surfaces

$$x^2 + y^2 + z^2 = \alpha x, \quad x^2 + y^2 + z^2 = \beta y, \quad x^2 + y^2 + z^2 = \gamma z$$

intersect at right angles.

15. Show that the normals to the surface $x = \varphi(u) \cos v$, $y = \varphi(u) \sin v$, $z = \psi(u)$ intersect the z-axis.

16. Find the surface formed by the normals to $y = x \tan z$ along the straight line $y = x$, $z = \frac{\pi}{4}$.

17. Find the equation of the osculating paraboloid to an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at the point $(0, 0, c)$.

18. Investigate the nature of points (viz., whether they are elliptic, hyperbolic or planar) on quadric surfaces.

19. Prove that if a smooth surface has only one point in common with a plane, then the plane is tangent at the point.

20. Prove that if a surface touches a plane along a certain line, then each point in the line is either parabolic or planar.

21. Let Φ be a surface, P a point on it, and α the tangent plane at P.

Prove that

(a) if P is elliptic, then all points on Φ, which are sufficiently near to P, are on one side of α,

(b) if P is hyperbolic, then there are points on Φ, as close to P as we please, and on opposite sides of α.

22. Prove that if all points on a curve γ on a surface are planar, then the curve is plane.

23. Prove that there are elliptic points on a closed surface.

24. Prove that if all normals to a surface intersect a certain straight line, then it is either a surface of revolution or a domain on a surface of revolution.

25. Prove that if all normals to a surface are concurrent, then it is either a sphere or a spherical domain.

Chapter XI

SURFACE CURVATURE

1. Surface Linear Element

Let Φ be an elementary surface obtained by a topological transformation of a domain G in the uv-plane, and $u = u(t)$, $v = v(t)$ a curve in G. A transformation of G into Φ transforms the curve into a curve γ on Φ. If Φ is given by a vector equation $\mathbf{r} = \mathbf{r}(u, v)$, then γ is specified by $\mathbf{r} = \mathbf{r}(u(t), v(t))$.

Its length is determined by the formula

$$s = \int \sqrt{\mathbf{r}_t'^2}\, dt = \int \sqrt{\mathbf{r}_u^2 \left(\frac{du}{dt}\right)^2 + 2\mathbf{r}_u\mathbf{r}_v \left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + \mathbf{r}_v^2 \left(\frac{dv}{dt}\right)^2}\, dt$$

$$= \int_\gamma \sqrt{\mathbf{r}_u^2\, du^2 + 2\mathbf{r}_u\mathbf{r}_v\, du\, dv + \mathbf{r}_v^2\, dv^2}, \qquad (*)$$

where $\int_\gamma$ denotes integration along γ.

The quadratic form

$$ds^2 = \mathbf{r}_u^2\, du^2 + 2\mathbf{r}_u\mathbf{r}_v\, du\, dv + \mathbf{r}_v^2\, dv^2$$

is called the *first fundamental form,* or *surface linear element.* We will employ the notation

$$\mathbf{r}_u^2 = E, \quad \mathbf{r}_u\mathbf{r}_v = F, \quad \mathbf{r}_v^2 = G$$

for its coefficients. It follows from the formulas (*) for the length of a curve, that, to measure it, the knowledge of its first fundamental form is sufficient, due to which the first fundamental form is said to *determine a metric* on the surface.

Let $u = u_1(t)$, $v = v_1(t)$ and $u = u_2(\tau)$, $v = v_2(\tau)$ be the equations of two curves in a domain G, which pass through a point (u_0, v_0). A transformation of G into a surface Φ carries them into two curves γ_1 and γ_2 on Φ. We call the angle θ between the half-tangents to these curves the angle between γ_1 and γ_2 at their common point $P(u_0, v_0)$.

We have

$$\cos\theta = \frac{\mathbf{r}_t'\mathbf{r}_\tau'}{\sqrt{\mathbf{r}_t'^2}\sqrt{\mathbf{r}_\tau'^2}} = \frac{(\mathbf{r}_u u_1' + \mathbf{r}_v v_1')(\mathbf{r}_u u_2' - \mathbf{r}_v v_2')}{\sqrt{(\mathbf{r}_u u_1' + \mathbf{r}_v v_1')^2}\sqrt{(\mathbf{r}_u u_2' + \mathbf{r}_v v_2')^2}}$$

$$= \frac{Eu_1'u_2' + F(u_1'v_2' + u_2'v_1') + Gv_1'v_2'}{(Eu_1'^2 + 2Fu_1'v_1' + Gv_1'^2)^{1/2}(Eu_2'^2 + 2Fu_2'v_2' + Gv_2'^2)^{1/2}}\,.$$

If we denote by d and δ differentiation with respect to u and v along γ_1 and γ_2, then the formula can be written as

$$\cos\theta = \frac{E\,du\,\delta u + F(du\,\delta v + dv\,\delta u) + G\,dv\,\delta v}{(E\,du^2 + 2F\,du\,dv + G\,dv^2)^{1/2}(E\,\delta u^2 + 2F\,\delta u\,\delta v + G\,\delta v^2)^{1/2}}\,. \qquad (**)$$

It is seen from (**) that *the angles between curves on a surface are also determined by the first fundamental form.*

We now clarify on what condition the u-, v-curves on the surface are orthogonal, i.e., intersect at right angles. Along the u-curves, $du \neq 0$, $dv = 0$, whereas along the v-curves $\delta v \neq 0$, $\delta u = 0$. Therefore, $\cos\theta = 0$, or *u-, v-curves are orthogonal if and only if $F\,du\,\delta v = 0$, i.e. if $F = 0$.*

A length-preserving, or, as we call it, metric-preserving transformation, is called a *deformation* of the surface. A deformation is also called an *isometric transformation.* Surfaces transformed into each other by an isometric transformation are said to be *isometric.* Under a suitable parametrization, isometric surfaces possess the same first fundamental form. A surface "in the small" is usually deformable. A surface "in the large", e.g., a sphere, may not be deformable. Any regular (twice differentiable) surface which is isometric to a sphere is a congruent sphere.

A transformation of a surface is said to be *conformal* if it is angle-preserving. Conformal transformations play an important part in cartography. Maps are actually conformal representations of domains on the earth's surface. The expedience of a map conformal representation is due to its similarity "in the small", and a faithful reproduction of the form of small domains.

2. Area of a Surface

Let Φ be a smooth surface. Partition it into small domains g, and take a point P as a base point in each, projecting one domain g onto the tangent plane at this point. Denote the projection area by $\sigma(g)$. By the *area* of Φ, we understand

$$S = \lim \sum_g \sigma(g),$$

provided that the domains g of the surface partition decrease in size without limit.

We now find the formula for the area of a surface given by a vector equation $\mathbf{r} = \mathbf{r}(u, v)$, for which we first derive an expression for $\sigma(g)$. Introduce Cartesian coordinates x, y, z by taking P as the origin, and the tangent plane at this point as the xy-plane. Let the surface be then given in g by equations

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v).$$

For sufficiently small g, their projections onto the tangent plane, i.e., the xy plane, are unique; therefore, u, v can be regarded as curvilinear coordinates on the projection. As is known from analysis, the area of a plane domain is found by the formula

$$\sigma = \iint \left\| \begin{matrix} x_u & y_u \\ x_v & y_v \end{matrix} \right\| du\, dv,$$

with respect to curvilinear coordinates.

The integrand can be represented in the form

$$\left\| \begin{matrix} x_u & y_u \\ x_v & y_v \end{matrix} \right\| = |(\mathbf{r}_u \wedge \mathbf{r}_v)\, \mathbf{n}_P|,$$

where $\mathbf{n}_P$ is the unit normal vector to the surface at P, and we can write

$$\sum_g \sigma(g) = \iint_\Phi |(\mathbf{r}_u \wedge \mathbf{r}_v)\, \mathbf{n}^*|\, du\, dv,$$

where $\mathbf{n}^*$ is a vector function on the surface, constant in each of the domains g, and equal to the unit vector of the normal at the base point P of the domain.

Now, passing to the limit, provided that g decrease in size without limits, we obtain the formula for the area

$$S = \iint_\Phi |(\mathbf{r}_u \wedge \mathbf{r}_v)\, \mathbf{n}|\, du\, dv.$$

Since the vectors $\mathbf{r}_u \wedge \mathbf{r}_v$ and $\mathbf{n}$ are collinear,

$$S = \iint_\Phi |\mathbf{r}_u \wedge \mathbf{r}_v|\, du\, dv.$$

Noticing that

$$|\mathbf{r}_u \wedge \mathbf{r}_v|^2 = \mathbf{r}_u^2 \mathbf{r}_v^2 - (\mathbf{r}_u \mathbf{r}_v)^2 = EG - F^2,$$

we obtain

$$S = \iint_\Phi \sqrt{EG - F^2}\, du\, dv.$$

We see that the *area of a surface, too, is determined by its first fundamental form.*

If a surface is given by an equation of the form $z = z(x, y)$, then

$$E = 1 + z_x^2, \quad F = z_x z_y, \quad G = 1 + z_y^2.$$

Therefore,

$$S = \int\int \sqrt{1 + z_x^2 + z_y^2}\, dx\, dy.$$

3. Normal Curvature of a Surface

Given a curve γ on a surface specified by a vector equation $\mathbf{r} = \mathbf{r}(u, v)$, we introduce the natural parameter (s) of this curve. Then u and v are functions of s, and the curve is given by an equation $\mathbf{r} = \mathbf{r}(u(s), v(s))$. As we know,

$$\mathbf{r}''_{ss} = k_1 \boldsymbol{\nu},$$

where $\boldsymbol{\nu}$ is the principal unit normal vector, and k_1 the curvature. Multiplying throughout by the unit surface normal vector $\mathbf{n}$, we obtain

$$\mathbf{r}''_{ss}\mathbf{n} = k_1 \cos\theta, \qquad (*)$$

θ being the angle between $\boldsymbol{\nu}$ and $\mathbf{n}$.

To transform the left-hand side, we see that

$$\mathbf{r}''_{ss} = \mathbf{r}_u u'' + \mathbf{r}_v v'' + \mathbf{r}_{uu} u'^2 + 2\mathbf{r}_{uv} u'v' + \mathbf{r}_{vv} v'^2.$$

Therefore,

$$\mathbf{r}''_{ss}\mathbf{n} = (\mathbf{r}_{uu}\mathbf{n})\, u'^2 + 2(\mathbf{r}_{uv}\mathbf{n})\, u'v' + (\mathbf{r}_{vv}\mathbf{n})\, v'^2$$
$$= \frac{(\mathbf{r}_{uu}\mathbf{n})\, du^2 + 2(\mathbf{r}_{uv}\mathbf{n})\, du\, dv + (\mathbf{r}_{vv}\mathbf{n})\, dv^2}{E\, du^2 + 2F\, du\, dv + G\, dv^2}.$$

The quadratic form in the numerator is called the *second fundamental form* of the surface. We will always use the notations

$$\mathbf{r}_{uu}\mathbf{n} = L, \quad \mathbf{r}_{uv}\mathbf{n} = M, \quad \mathbf{r}_{vv}\mathbf{n} = N$$

for its coefficients.

Now, we derive from (*) that

$$k_1 \cos\theta = \frac{L\, du^2 + 2M\, du\, dv + N\, dv^2}{E\, du^2 + 2F\, du\, dv + G\, dv^2}.$$

Hence, $k_1 \cos\theta$ depends only on the direction of γ, i.e., the value du/dv. Therefore, $k_1 \cos\theta$ is the same for all curves with a common tangent. If we take as the curve the surface section by a plane perpendicular to the tangent plane (*normal section*), then $|\cos\theta| = 1$; consequently,

$$k_1 |\cos\theta| = k_0,$$

where k_0 is the normal section curvature. If we ascribe a suitable sign to normal curvature, then the formula can be simply written as

$$k_1 \cos\theta = k_0. \qquad (**)$$

Meanwhile,

$$k_0=\frac{L\,du^2+2M\,du\,dv+N\,dv^2}{E\,du^2+2F\,du\,dv+G\,dv^2}$$

The relation (**) between curvature on a surface and normal curvature is the subject matter of the *Meusnier theorem.*

We now obtain expressions for the first and second fundamental form coefficients if the surface is given by parametric equations

$$x=x\,(u,\ v),\quad y=y\,(u,\ v),\quad z=z\,(u,\ v).$$

We have

$$E=\mathbf{r}_u^2=x_u^2+y_u^2+z_u^2,$$
$$F=\mathbf{r}_u\mathbf{r}_v=x_ux_v+y_uy_v+z_uz_v,$$
$$G=\mathbf{r}_v^2=x_v^2+y_v^2+z_v^2,$$

and

$$L=\mathbf{r}_{uu}\mathbf{n}=\mathbf{r}_{uu}\frac{(\mathbf{r}_u\wedge\mathbf{r}_v)}{|\mathbf{r}_u\wedge\mathbf{r}_v|}=\frac{(\mathbf{r}_{uu}\mathbf{r}_u\mathbf{r}_v)}{|\mathbf{r}_u\wedge\mathbf{r}_v|}=\frac{\begin{vmatrix} x_{uu} & y_{uu} & z_{uu}\\ x_u & y_u & z_u\\ x_v & y_v & z_v\end{vmatrix}}{\sqrt{EG-F^2}}.$$

Similarly, we find

$$M=\frac{\begin{vmatrix} x_{uv} & y_{uv} & z_{uv}\\ x_u & y_u & z_u\\ x_v & y_v & z_v\end{vmatrix}}{\sqrt{EG-F^2}},\quad N=\frac{\begin{vmatrix} x_{vv} & y_{vv} & z_{vv}\\ x_u & y_u & z_u\\ x_v & y_v & z_v\end{vmatrix}}{\sqrt{EG-F^2}}.$$

To determine the coefficients of fundamental forms of the surface if it is given by an equation such as $z=z\,(x,\ y)$, it suffices to notice that the specification is equivalent to parametric equations

$$x=u,\quad y=v,\quad z=z\,(u,\ v),$$

in which case we obtain for the coefficients,

$$E=1+p^2,\quad F=pq,\quad G=1+q^2,$$
$$L=\frac{r}{\sqrt{1+p^2+q^2}},\quad M=\frac{s}{\sqrt{1+p^2+q^2}},\quad N=\frac{t}{\sqrt{1+p^2+q^2}},$$

where p, q, r, s, t are the first and second derivatives of the function $z\,(x,\ y)$, viz., $p=z_x$, $q=z_y$, $r=z_{xx}$, $s=z_{xy}$, $t=z_{yy}$.

4. Indicatrix of the Normal Curvature

Take a point O on a surface as the origin of coordinates, and the tangent plane in it as the xy-plane. As we know, the surface in the neighbourhood of O is then given by the equation

$$z=\frac{1}{2}\,(rx^2+2sxy+ty^2)+\varepsilon\,(x,\ y)\,(x^2+y^2),$$

where $\varepsilon\,(x,\ y)\to 0$ as $x,\ y\to 0$.

The osculating paraboloid at O is

$$z=\frac{1}{2}(rx^2+2sxy+ty^2),$$

whereas the Dupin indicatrix at the same point is

$$rx^2+2sxy+ty^2=\pm 1.$$

It is easy to see that the first and second fundamental forms of the surface, and those of the osculating paraboloid at O, are the same. Viz., the first and second fundamental forms are

$$dx^2+dy^2$$

and

$$r\,dx^2+2s\,dx\,dy+t\,dy^2.$$

Hence, *the normal curvature of a surface and its osculating paraboloid is the same in the same direction.* Viz.,

$$k_n=\frac{r\,dx^2+2s\,dx\,dy+t\,dy^2}{dx^2+dy^2}.$$

We now turn to the Dupin indicatrix at O (Fig. 101), and find the expression for the normal curvature in a direction OQ in terms of the coordinates x and y of the point Q in the indicatrix. We have $dx : dy = x : y$. Therefore,

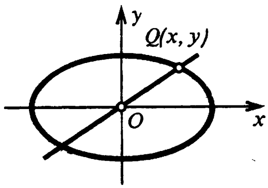

Fig. 101

$$k_n=\frac{rx^2+2sxy+ty^2}{x^2+y^2}.$$

Since Q is in the indicatrix, the numerator equals ± 1, and the denominator OQ^2. Hence,

$$k_n=\frac{\pm 1}{OQ^2} \qquad (*)$$

which reveals the relation of the Dupin indicatrix to the normal curvature, and, therefore, the origin of its second name, "the indicatrix of the normal curvature".

From the formula (*), we derive that

(i) *the normal curvature of a surface in an asymptotic direction is zero*, and

(ii) *the normal curvature of a surface along principal directions attains extreme values.*

We now take the principal directions as those of the coordinate axes x and y. Then $s = 0$, and

$$k_n=r\left(\frac{dx}{\sqrt{dx^2+dy^2}}\right)^2+t\left(\frac{dy}{\sqrt{dx^2+dy^2}}\right)^2$$

Putting $dy = 0$, we obtain $k_n' = r$, while setting $dx = 0$, $k_n'' = t$, where k_n'' and k_n' are the normal curvatures along the principal directions.

Denoting

$$dx/\sqrt{dx^2+dy^2}=\cos\theta, \quad dy/\sqrt{dx^2+dy^2}=\sin\theta,$$

we obtain the *Euler formula*

$$k_n = k_n' \cos^2\theta + k_n'' \sin^2\theta, \qquad (**)$$

θ being the angle made by a given direction with the principal which is associated with k_n'.

5. Conjugate Coordinate Lines on a Surface

The above concept of conjugate directions of a surface is related to the Dupin indicatrix, due to which we find the equations of the osculating paraboloid and Dupin indicatrix in an oblique coordinate system x, y, z closely related to a parametrization of u, v. Viz., we take $\mathbf{r}_u$, $\mathbf{r}_v$ and $\mathbf{n}$ as the basis vectors along the coordinate axes. Let a point O on the surface be associated with coordinates $u = u_0$, $v = v_0$. The surface equation in the neighbourhood of O can be written as

$$\mathbf{r}=\mathbf{r}_u(u-u_0)+\mathbf{r}_v(v-v_0)+\frac{1}{2}(\mathbf{r}_{uu}(u-u_0)^2+2\mathbf{r}_{uv}(u-u_0)(v-v_0)$$
$$+\mathbf{r}_{vv}(v-v_0)^2)+\varepsilon(u,v)[(u-u_0)^2+(v-v_0)^2].$$

We assert that the paraboloid given by the equation

$$z=\frac{1}{2}(Lx^2+2Mxy+Ny^2) \qquad (*)$$

is osculating at O. In fact, it can be given by parametric equations

$$x=u-u_0, \quad y=v-v_0,$$
$$z=\frac{1}{2}(L(u-u_0)^2+2M(u-u_0)(v-v_0)+N(v-v_0)^2).$$

It can be easily verified by computation that the paraboloid and surface at O have the same first and second fundamental forms and, therefore, the same normal curvature, which already fully determines an osculating paraboloid.

From the osculating paraboloid equation (*), we obtain that of the indicatrix of the normal curvature,

$$Lx^2 + 2Mxy + Ny^2 = \pm 1.$$

As we know, for two directions dx/dy and $\delta x/\delta y$ to be conjugate with respect to this curve, it is sufficient that

$$L\,dx\,\delta x + 2M(dx\,\delta y + dy\,\delta x) + N\,dy\,\delta y = 0.$$

Since $dx = du$, $dy = dv$, $\delta x = \delta u$, $\delta y = \delta v$ at O, the condition for conjugacy of the directions d and δ of the surface is

$$L\, du\, \delta u + 2M\, (du\, \delta v + dv\, \delta u) + N\, dv\, \delta v = 0.$$

We call u- and v-curves *conjugate* if the coordinate line directions at each point are conjugate. *In the case of conjugate lines*, $M = 0$. Conversely, *if* $M = 0$, then *the coordinate lines are conjugate*. Indeed, $dv = 0$ in the direction of the u-curves, and $\delta u = 0$ in that of the v-curves. Therefore, $2M du\, \delta v = 0$, with the consequence that $M = 0$. Conversely, if $M = 0$, then $2M du\, \delta v = 0$.

A line in a surface is said to be *asymptotic* if its direction at each point is asymptotic. Since the normal curvature along an asymptotic direction is zero,

$$L\, du^2 + 2M\, du\, dv + N\, dv^2 = 0,$$

which is just the *asymptotic line equation*.

If the coordinate lines on a surface are asymptotic, then $L = 0$, *and* $N = 0$. Conversely, *if* $L = 0$, *and* $N = 0$, *then the coordinate lines are asymptotic*.

In fact, if a u-curve is asymptotic, then $L\, du^2 = 0$, and $L = 0$. If a v-curve is asymptotic, then $N dv^2 = 0$, and $N = 0$. Conversely, if $L = 0$, and $N = 0$, then $L du^2 = 0$, and $N dv^2 = 0$, i.e., the coordinate lines are asymptotic.

Due to the second fundamental form simplicity in the case of asymptotic coordinate lines, it seems expedient to make use of the latter in general considerations. However, it should be borne in mind that asymptotic coordinate lines can be introduced only in the neighbourhood of a hyperbolic point, whereas conjugate ones in the neighbourhood of an elliptic or a hyperbolic point, and an arbitrary family of coordinate lines can be taken, provided they have no asymptotic directions.

Remark. The concept of asymptotic direction has been defined by us in terms of the Dupin indicatrix, and related only to the case of a hyperbolic point. Meanwhile, it is completely characterized by the fact that the normal curvature along this direction was zero, due to which we can extend the notion of asymptotic direction to the cases of parabolic and planar points, assuming a direction asymptotic if the normal curvature is zero. With this definition, we still have two asymptotic directions at a hyperbolic point, and one at a parabolic point, whereas, at a planar point, any direction is asymptotic.

6. Lines of Curvature

The principal directions of a surface have been defined by us as those of the Dupin indicatrix axes. We then proved (see Sec. 4) that principal directions are characterized by the normal curvature having extreme values along them. Therefore, principal directions can be specified just by this property, and the concept of principal direction is then extendable to planar points with no Dupin indicatrix. Since the normal curvature is zero at a planar point in any direction, each direction is principal.

Generally speaking, there are two principal directions at each point of a surface, with the exception of planar and special elliptic points with a circle as the Dupin indicatrix (*spherical points*), where any direction is principal.

We now find a condition on which a direction du/dv of a surface should be principal.

We have

$$k_n = \frac{\mathrm{II}}{\mathrm{I}} = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dn^2}\,.$$

Since the right-hand side has an extreme value for a principal direction as a function of du, dv, its derivatives with respect to these variables are zeros.

Hence,

$$\frac{2(L\,du + M\,dv)}{\mathrm{I}} - \frac{2(E\,du + F\,dv)}{\mathrm{I}^2}\,\mathrm{II} = 0,$$

$$\frac{2(M\,du + N\,dv)}{\mathrm{I}} - \frac{2(F\,du + G\,dv)}{\mathrm{I}^2}\,\mathrm{II} = 0,$$

where I and II denote the first and second fundamental forms.

We derive

$$\frac{L\,du + M\,dv}{E\,du + F\,dv} = \frac{\mathrm{II}}{\mathrm{I}} = k_n,$$

$$\frac{M\,du + N\,dv}{F\,du + G\,dv} = \frac{\mathrm{II}}{\mathrm{I}} = k_n.$$

Consequently, the principal direction equation is

$$\frac{L\,du + M\,dv}{E\,du + F\,dv} - \frac{M\,du + N\,dv}{F\,du + G\,dv} = 0,$$

which can be written in a form more convenient to be committed to memory, viz.,

$$\begin{vmatrix} dv^2 & -du\,dv & du^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0. \qquad (*)$$

A line on a surface is called a *line of curvature* if its direction is principal at each point. Therefore, (*) is the *curvature line differential equation.*

If the coordinate lines on a surface are those of curvature in a domain containing no planar or spherical points, then $F = 0$, and

$M = 0$. Indeed, there are two principal, orthogonal and conjugate directions at each point. Therefore, $F = 0$, and $M = 0$.

In conclusion, we prove the following *Rodrigues theorem.*

In differentiating along a principal direction,

$$d\mathbf{n} = -k_n\, d\mathbf{r},$$

where k_n is the normal curvature.

Proof. Introduce coordinate lines u, v so that the direction of the u-curve at a given point is principal, and the coordinate lines are orthogonal. Since $\mathbf{n}^2 = 1$, we have $\mathbf{n}_u\mathbf{n} = 0$, i.e., the vector $\mathbf{n}_u$ is perpendicular to $\mathbf{n}$, and, therefore, admits a resolution in terms of the vectors $\mathbf{r}_u$ and $\mathbf{r}_v$, viz.,

$$\mathbf{n}_u = \lambda\mathbf{r}_u + \mu\mathbf{r}_v.$$

Multiplying scalarly throughout by $\mathbf{r}_v$, and noticing that $\mathbf{r}_u\mathbf{r}_v = 0$ (*orthogonality*), $\mathbf{n}_u\mathbf{r}_v = -M = 0$ (*conjugacy*), we obtain $\mu = 0$. Now, multiplying throughout by $\mathbf{r}_u$, we get

$$\mathbf{n}_u\mathbf{r}_u = \lambda\mathbf{r}_u^2, \quad \text{i.e.,} \quad -L = \lambda E.$$

Hence,

$$-\lambda = \frac{L}{E},$$

but this is the normal curvature k_n along the direction of u.

Thus,

$$\mathbf{n}_u = -k_u\mathbf{r}_u.$$

Q.E.D.

7. Mean and Gaussian Curvature of a Surface

The *mean curvature* of a surface is half the sum of the principal curvatures. The *total*, or *Gaussian*, curvature of a surface is the product of the principal curvatures.

At an elliptic point, the principal curvatures have like signs; therefore, the Gaussian curvature is positive. At a hyperbolic point, the principal curvatures have unlike signs; therefore, the Gaussian curvature is negative. At a parabolic or planar point, the Gaussian curvature is zero.

We now find the expression for the mean and Gaussian curvature of a surface in terms of the first and second fundamental form coefficients. In the previous section, we have derived two formulas for the normal curvature along a principal direction du/dv, viz.,

$$k_n = \frac{L\,du + M\,dv}{E\,du + F\,dv}, \quad k_n = \frac{M\,du + N\,dv}{F\,du + G\,dv},$$

which can be rewritten as

$$L\,du + M\,dv - k_n(E\,du + F\,dv) = 0,$$
$$M\,du + N\,dv - k_n(F\,du + G\,dv) = 0.$$

Eliminating du and dv, we obtain

$$\begin{vmatrix} L-Ek_n & M-Fk_n \\ M-Fk_n & N-Gk_n \end{vmatrix}=0,$$

or

$$(EG-F^2)\,k_n^2-(LG-2FM+NE)\,k_n+(LN-M^2)=0.$$

This quadratic equation has two roots, k_n' and k_n'', the principal curvatures of the surface.

By the property of the roots of a quadratic equation,

$$\frac{k_n'+k_n''}{2}=\frac{LG-2FM+NE}{2\,(EG-F^2)},$$

$$k_n'k_n''=\frac{LN-M^2}{EG-F^2}.$$

Such are the expressions for mean and Gaussian curvature.

The concept of total curvature was introduced by F. Gauss who gave another definition. Viz., let P be an arbitrary point of a surface, and g its small neighbourhood. Translate the unit normal vectors at different points of the domain g, so that they have a common origin. Then their ends are on the unit sphere, and form a certain set $\bar{g}$ (*spherical image* of g). According to F. Gauss, the total curvature of the surface at P is the limit of the ratio of the area of $\bar{g}$ to that of g as g is contracted to P. We show that this definition leads to the same expression, i.e., the principal curvature product. For simplicity, we confine ourselves to the case of an elliptic point P.

We introduce coordinate lines u, v in the neighbourhood of P, so that their directions are principal at the point.

The domain area is

$$S(g)=\int\int|\mathbf{r}_u\wedge\mathbf{r}_v|\,du\,dv,$$

whereas that of $\bar{g}$

$$S(\bar{g})=\int\int|\mathbf{n}_u\wedge\mathbf{n}_v|\,du\,dv.$$

Since the domain of integration with respect to the variables u, v is the same in both formulas,

$$\lim_{g\to P}\frac{S(\bar{g})}{S(g)}=\frac{|\mathbf{n}_u\wedge\mathbf{n}_v|}{|\mathbf{r}_u\wedge\mathbf{r}_v|}$$

By the Rodrigues theorem, $\mathbf{n}_u=-k_n'\mathbf{r}_u$, $\mathbf{n}_v=-k_n''\mathbf{r}_v$. Therefore,

$$\lim_{g\to P}\frac{S(\bar{g})}{S(g)}=k_n'k_n''.$$

Q.E.D.

8. Example of a Surface of Constant Negative Gaussian Curvature

An example of a surface of zero Gaussian curvature is the plane. Its normal curvature along any direction is zero. Therefore, the Gaussian curvature is also zero.

An example of a surface of constant positive curvature is a sphere with radius R. Its normal curvature along any direction is $1/R$. Therefore, the Gaussian curvature is $1/R^2$.

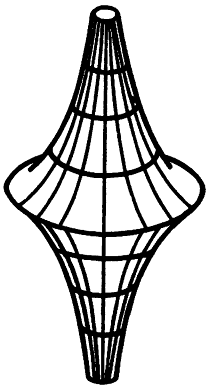

Fig. 102

We now construct an example of a surface of constant negative Gaussian curvature. We shall seek it among surfaces of revolution.

A *surface of revolution* is obtained by rotating a plane curve about an axis in the plane. Sections of a surface of revolution by planes passing through the axis are called *meridians*, and those by planes perpendicular to the axis *parallels*.

Since a surface of revolution is symmetric about the plane of any meridian, its directions along meridians are principal. Therefore, directions along parallels are also principal.

It is obvious that the normal curvature of a surface along the direction of a meridian is the curvature of the latter. The normal curvature along the direction of a parallel is expressed in terms of its curvature by the Meusnier formula.

Take as the z-axis that of the surface, and consider a meridian in the xz-plane. Let its equation be $x = x(z)$. The normal curvature along its direction is then

$$k_n' = \frac{x''}{(1+x'^2)^{3/2}},$$

whereas that along the parallel

$$k_n'' = -\frac{1}{x(1+x'^2)^{1/2}},$$

$1/x$ being curvature, and $1/(1+x'^2)^{1/2}$ the cosine of the angle between the tangent to the meridian and the axis of the surface (z-axis). Hence, the Gaussian curvature is

$$K = k_n' k_n'' = -\frac{x''}{x(1+x'^2)^2}.$$

Multiplying throughout by xx', we obtain

$$Kxx' = -\frac{x'x''}{(1+x'^2)^2};$$

integrating, we get

$$Kx^2 + c = \frac{1}{1+x'^2},$$

where c is a constant. To make further integration in terms of elementary functions possible, we put $c = 1$.

Then

$$Kx^2 = -\frac{x'^2}{1+x'^2}.$$

Set $x' = \tan\theta$. We have

$$Kx^2 = -\sin^2\theta, \quad x = \frac{1}{\sqrt{-K}}\sin\theta.$$

Further,

$$\frac{dz}{dx} = \cot\theta, \quad dz = \frac{1}{\sqrt{-K}}\frac{\cos^2\theta}{\sin\theta}\,d\theta = \frac{1}{\sqrt{-K}}\left(\frac{1}{\sin\theta} - \sin\theta\right)d\theta.$$

Hence,

$$z = \frac{1}{\sqrt{-K}}\left(\cos\theta + \ln\tan\frac{\theta}{2}\right) + c_1.$$

The constant c_1 can be assumed to be equal to zero. Thus, the meridian can be given parametrically as

$$x = \frac{1}{\sqrt{-K}}\sin\theta, \quad z = \frac{1}{\sqrt{-K}}\left(\cos\theta + \ln\tan\frac{\theta}{2}\right).$$

The curve is called a *tractrix*, and the surface of constant negative curvature obtained by rotation around the z-axis a *pseudosphere* (Fig. 102).

EXERCISES TO CHAPTER XI

1. Find the first fundamental form of the surface of revolution $x = \varphi(u)\cos v$, $y = \varphi(u)\sin v$, $z = \psi(u)$.
2. Show that a surface of revolution can be parametrized so that its first fundamental form is $du^2 + G(u)\,dv^2$.
3. Find the length of the curve given by $u = v$ on the surface whose first fundamental form is $du^2 + \sinh^2 u\,dv^2$.
4. Find the angle at which the coordinate lines $x = x_0$, $y = y_0$ intersect on the surface $z = axy$.
5. Show that the coordinate lines u, v are orthogonal on the helicoid $x = au\cos v$, $y = au\sin v$, $z = bv$.
6. Find the curves (called *loxodromes*) making equal angles with the meridians on a sphere.
7. Find the area of the quadrilateral bounded by $u = 0$, $u = 1$,

$v = 0$, $v = 1$ on the helicoid

$$x = u \cos v, \ y = u \sin v, \ z = v.$$

8. Show that the areas of domains on the paraboloids $z = \frac{a}{2}(x^2 + y^2)$, $z = axy$, projected onto the same domain of the xy-plane, are equal.

9. Show that if a surface admits a parametrization such that the first fundamental form coefficients are independent of u and v, then the surface is locally isometric to the plane.

10. Prove that there exists a conformal mapping of a surface of revolution onto a plane, so that the surface meridians are carried into straight lines passing through the origin, and the parallels into circles centred at the origin. Consider the particular case of

$$x = \cos u \cos v, \ y = \cos u \sin v, \ z = \sin u \ (\textit{sphere}).$$

11. Prove that there exists a conformal mapping of a sphere onto a plane such that the meridians and parallels are sent into straight lines $x = \text{const}$ and $y = \text{const}$.

12. Show that there is an isometric mapping of a helicoid $x = u \cos v$, $y = u \sin v$, $z = mv$ onto a catenoid $x = \alpha \cos \beta$, $y = \alpha \sin \beta$, $z = m \cosh^{-1} \frac{\alpha}{m}$, so that the rectilinear generators of the former correspond to the meridians of the latter.

13. Find the second fundamental form of the helix $x = u \cos v$, $y = u \sin v$, $z = v$.

14. Find the normal curvature of a paraboloid $z = \frac{1}{2}(ax^2 + by^2)$ at the point (0, 0) along the direction $dx{:}dy$.

15. Show that, for any parametrization, the second fundamental form of the plane is identically zero and directly proportional to the first fundamental form under any parametrization of a sphere.

16. Find asymptotic lines on the surface $z = \frac{x}{y} + \frac{y}{x}$.

17. Find asymptotic lines on the catenoid

$$x = \cosh u \cos v, \quad y = \cosh u \sin v, \ z = u.$$

18. Show that one family of asymptotics on a helicoid consists of straight lines, and the other of helices.

19. Prove that the coordinate u-, v-curves on a translation surface $\mathbf{r} = U(u) + V(v)$ are conjugate.

20. Show that meridians and parallels of a surface of revolution are its lines of curvature.

21. Determine the principal curvatures of a paraboloid $z = axy$ at the point (0, 0, 0).

22. Find the lines of curvature on the helicoid

$$x = u \cos v, \quad y = u \sin v, \quad z = cv.$$

23. Find the mean and Gaussian curvatures of a paraboloid $z = axy$ at the point (0, 0, 0).

24. Prove that Gaussian curvature is positive at elliptic points of a surface, negative at hyperbolic, and zero at parabolic and planar.

25. Show that the mean curvature of a helicoid and catenoid is zero.

26. Prove that the Gaussian curvature of a cylindrical or conical surface is zero.

27. Prove that the Gaussian curvature of the surface formed by the tangents to a curve is zero.

28. Show that if the mean curvature of a surface is zero everywhere, then the asymptotic lines are orthogonal.

29. Show that if each point on a surface is spherical, i.e., the normal curvature along any direction is unaltered, then the surface is either a sphere or part of a sphere.

30. A surface Φ is said to be *parallel* to a surface F if it is the locus of the ends of line segments of constant length, cut off on the normals to F. Their ends are regarded as corresponding points of the surfaces.

Show that

(a) the tangent planes at the corresponding points of F and Φ are parallel,

and

(b) the lines of curvature of F correspond to those of Φ.

31. Express the mean and Gaussian curvature of a surface in terms of those of a parallel surface.

32. Prove that a spherical mapping of a surface of zero mean curvature is conformal.

Chapter XII

INTRINSIC GEOMETRY OF SURFACE

1. Gaussian Curvature as an Object of the Intrinsic Geometry of Surfaces

By the *intrinsic geometry* of a surface, we understand the branch of geometry, which studies the properties of the surface and its figures in relation only to the length of curves.

As to regular surfaces, we can say that their intrinsic geometry studies properties determined by the first fundamental form. Thus, the length of curves on a surface, angles between them and areas of domains are objects of the intrinsic geometry. We shall now prove that *Gaussian curvature is also an object of intrinsic geometry*, since it admits en expression in terms of the first fundamental form coefficients only.

We have

$$K=\frac{LN-M^2}{EG-F^2},$$

$$LN=\frac{1}{EG-F^2}\begin{vmatrix} x_{uu} & y_{uu} & z_{uu}\\ x_u & y_u & z_u\\ x_v & y_v & z_v\end{vmatrix}\cdot\begin{vmatrix} x_{vv} & y_{vv} & z_{vv}\\ x_u & y_u & z_u\\ x_v & y_v & z_v\end{vmatrix}.$$

Multiplying the determinants together according to the familiar rules, we obtain

$$LN=\frac{1}{EG-F^2}\begin{vmatrix} (\mathbf{r}_{uu}\mathbf{r}_{vv}) & (\mathbf{r}_{uu}\mathbf{r}_u) & (\mathbf{r}_{uu}\mathbf{r}_v)\\ (\mathbf{r}_u\mathbf{r}_{vv}) & E & F\\ (\mathbf{r}_v\mathbf{r}_{vv}) & F & G\end{vmatrix}.$$

Similarly,

$$M^2=\frac{1}{EG-F^2}\begin{vmatrix} (\mathbf{r}_{uv})^2 & (\mathbf{r}_{uv}\mathbf{r}_u) & (\mathbf{r}_{uv}\mathbf{r}_v)\\ (\mathbf{r}_u\mathbf{r}_{uv}) & E & F\\ (\mathbf{r}_v\mathbf{r}_{uv}) & F & G\end{vmatrix}.$$

Hence,

$$K=\frac{1}{EG-F^2}\left\{\begin{vmatrix} (\mathbf{r}_{uu}\mathbf{r}_{vv})-(\mathbf{r}_{uv})^2 & (\mathbf{r}_{uu}\mathbf{r}_u) & (\mathbf{r}_{uu}\mathbf{r}_v)\\ (\mathbf{r}_u\mathbf{r}_{vv}) & E & F\\ (\mathbf{r}_v\mathbf{r}_{vv}) & F & G\end{vmatrix} - \begin{vmatrix} 0 & (\mathbf{r}_{uv}\mathbf{r}_u) & (\mathbf{r}_{uv}\mathbf{r}_v)\\ (\mathbf{r}_u\mathbf{r}_{uv}) & E & F\\ (\mathbf{r}_v\mathbf{r}_{uv}) & F & G\end{vmatrix}\right\}.$$

Differentiating

$$\mathbf{r}_u^2=E,\quad \mathbf{r}_u\mathbf{r}_v=F,\quad \mathbf{r}_v^2=G$$

with respect to u and v, we obtain

$$\mathbf{r}_{uu}\mathbf{r}_u=\frac{1}{2}E_u,\quad \mathbf{r}_{uv}\mathbf{r}_v=\frac{1}{2}G_u,$$

$$\mathbf{r}_{uv}\mathbf{r}_u=\frac{1}{2}E_v,\quad \mathbf{r}_{uu}\mathbf{r}_v=F_u-\frac{1}{2}E_v,$$

$$\mathbf{r}_{vv}\mathbf{r}_v=\frac{1}{2}G_v,\quad \mathbf{r}_{vv}\mathbf{r}_u=F_v-\frac{1}{2}G_u.$$

Now, differentiating the fifth equality with respect to v, the fourth with respect to u, and subtracting termwise, we obtain

$$\mathbf{r}_{uu}\mathbf{r}_{vv}-\mathbf{r}_{uv}^2=-\frac{1}{2}G_{uu}+F_{uv}-\frac{1}{2}E_{vv}.$$

Substituting the values found in the expression for Gaussian curvature, we derive

$$K=\frac{1}{EG-F^2}\left\{\begin{vmatrix}\left(-\frac{1}{2}G_{uu}+F_{uv}-\frac{1}{2}E_{vv}\right) & \frac{1}{2}E_u & \left(F_u-\frac{1}{2}E_v\right)\\ \left(F_v-\frac{1}{2}G_u\right) & E & F\\ \frac{1}{2}G_u & F & G\end{vmatrix}-\begin{vmatrix}0 & \frac{1}{2}E_v & \frac{1}{2}G_u\\ \frac{1}{2}E_v & E & F\\ \frac{1}{2}G_u & F & G\end{vmatrix}\right\}.$$

It was F. Gauss who for the first time expressed total curvature only in terms of the first fundamental form coefficients and their derivatives.

Note that if a surface is parametrized so that its first fundamental form is

$$ds^2 = du^2 + G\,dv^2,$$

then the Gaussian curvature is

$$K=-\frac{1}{\sqrt{G}}(\sqrt{G})_{uu},$$

to see which, it suffices to make use of the above (Gauss) formula.

The expression of Gaussian curvature only in terms of the first fundamental form coefficients and their derivatives demonstrates that the first and second fundamental forms of a surface are not independent. The question arises naturally whether there are other relations between the coefficients. Another two formulas obtained by K. M. Peterson and D. Codazzi turn out to be valid, viz.,

$$2(EG-F^2)(L_v-M_u)-(EN-2FM+GL)(E_v-F_u)+\begin{vmatrix}E & E_u & L\\ F & F_u & M\\ G & G_u & N\end{vmatrix}=0,$$

$$2(EG-F^2)(M_v-N_u)-(EN-2FM+GL)(F_v-G_u)+\begin{vmatrix}E & E_v & L\\ F & F_v & M\\ G & G_v & N\end{vmatrix}=0.$$

The following *Bonnet theorem* states that there are no other relations between the first and second fundamental form coefficients.

Let

$$E\,du^2 + 2F\,du\,dv + G\,dv^2 \text{ and } L\,du^2 + 2M\,du\,dv + N\,dv^2$$

be any two quadratic forms, of which the former is positive definite. If the Gauss-Peterson-Codazzi relations hold for their coefficients, then there exists, and is unique up to disposition in space, a surface for which these are the first and second fundamentals forms, respectively.

2. Geodesic Lines on a Surface

A line on a surface is said to be *geodesic* if its principal normal at each point where the curvature is other than zero coincides with the normal to the surface.

We now make up the differential equation for geodesics. Let $\mathbf{r} = \mathbf{r}(t)$ be any parametrization of a geodesic. Since the vectors $\mathbf{r}'$ and $\mathbf{r}''$ lie in the osculating plane,

$$(\mathbf{r}''\mathbf{r}'\mathbf{n}) = 0. \qquad (*)$$

We can always locally take u or v as a parameter of the line. If we take u, then

$$\mathbf{r}' = \mathbf{r}_u + \mathbf{r}_v v'$$

and

$$\mathbf{r}'' = \mathbf{r}_{uu} + 2\mathbf{r}_{uv}v' + \mathbf{r}_{vv}v'^2 + \mathbf{r}_v v''.$$

Substituting these expressions in $(*)$, and solving the equation for v'', we get

$$v'' = \frac{1}{(\mathbf{r}_u\mathbf{r}_v\mathbf{n})}\,(\mathbf{r}_{uu} + 2\mathbf{r}_{uv}v' + \mathbf{r}_{vv}v'^2\mathbf{r}_u + \mathbf{r}_v v'\mathbf{n})$$

which is seen to be a second-order differential equation. It follows from the unique existence theorem for solutions to such an equation that, *along any direction, one, and only one, geodesic passes through each point of the surface.*

It is obvious that straight lines on a plane are geodesics. Since a straight line can be drawn through any point in a plane and along any direction, they exhaust all the plane geodesics. Similarly, great circles, and they only, are geodesics on a sphere.

A parametrization of a surface is said to be *semi-geodesic* if the coordinate lines of one family are geodesics, and those of the other are orthogonal to the former. We now clarify what is the form of the surface linear element with respect to such a semi-geodesic parametrization. E.g., let a family of u-curves consist of geodesic lines. Then

$$(\mathbf{r}_{uu}\mathbf{r}_u\mathbf{n}) = 0. \qquad (**)$$

Resolve $\mathbf{r}_{uu}$ in terms of non-coplanar vectors $\mathbf{r}_u$, $\mathbf{r}_v$ and $\mathbf{n}$. We have

$$\mathbf{r}_{uu} = \alpha\mathbf{r}_u + \beta\mathbf{r}_v + \gamma\mathbf{n}. \qquad (***)$$

Substituting this in (**), we obtain $\beta\,(\mathbf{r}_u\mathbf{r}_v\mathbf{n}) = 0$, i.e., $\beta = 0$.
Multiplying (***) scalarly by $\mathbf{r}_v$, and noticing that $\mathbf{r}_u\mathbf{r}_v = F = 0$ (net being orthogonal), we obtain

$$\mathbf{r}_{uu}\mathbf{r}_v = \beta\mathbf{r}_v^2 = 0,$$

whereas

$$\mathbf{r}_{uu}\mathbf{r}_v = (\mathbf{r}_u\mathbf{r}_v)_u - \frac{1}{2}(\mathbf{r}_u^2)_v = -\frac{1}{2}E_v = 0.$$

Therefore, E depends only on u.
We introduce a new parameter $\bar{u}$ by putting

$$d\bar{u} = \sqrt{E}\,du.$$

The linear element

$$ds^2 = E\,du^2 + G\,dv^2$$

then takes the form

$$ds^2 = d\bar{u}^2 + G\,dv^2.$$

In turns out that a semi-geodesic parametrization of a surface can be introduced always, and very much at random. Viz., *if γ is a curve on a surface, then we can introduce such a semi-geodesic parametrization in its neighbourhood that one family of coordinate lines consists of geodesics orthogonal to γ.* However, we do not give the proof.

3. Extremal Property of Geodesics

Here, we prove the following extremal property of geodesics.
A geodesic on a sufficiently small line segment is shorter than any curve near to it, which passes through the same points.

Proof. Let γ be a geodesic, P a point in it, and A, B two of its points near to P. We prove that any curve joining A and B, which is near to γ, will be longer than the line segment AB on γ.

Draw through P a geodesic $\bar{\gamma}$ perpendicular to γ, and introduce a semi-geodesic parametrization in the neighbourhood of the point by taking geodesics orthogonal to $\bar{\gamma}$ as the family of u-curves.

Let $\tilde{\gamma}$ be any curve joining A and B in the parametrized neighbourhood. Then its length is

$$s = \int_{(A)}^{(B)} \sqrt{du^2 + G\,dv^2} > \int_{(A)}^{(B)} |du| \geqslant |u(B) - u(A)|,$$

where $A\ |\,u(B) - u(A)\,|$ is that of the line segment AB of γ.

Q.E.D.

The geodesics' extremal property permits us to obtain their equations as those of the variational problem for the functional

$$s = \int \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}\,dt,$$

containing only the first fundamental form coefficients E, F, G and their derivatives, which means that *geodesics are an object of the intrinsic geometry of the surface.*

4. Surfaces of Constant Gaussian Curvature

Let Φ be a surface of constant Gaussian curvature K, and P a point on it. Draw an arbitrary geodesic $\bar{\gamma}$ through P, and introduce a semi-geodesic parametrization in its neighbourhood by taking the geodesics orthogonal to $\bar{\gamma}$ as the family of u-curves. The surface linear element then assumes the form

$$ds^2 = du^2 + G\,dv^2.$$

We take arc length along $\bar{\gamma}$ as the parameter v. Therefore, $G(0, v) = 1$ along it, i.e., when $u = 0$.

We show that then $G_u = 0$ on $\bar{\gamma}$. Since $\bar{\gamma}$ is a geodesic, $(\mathbf{r}_{vv}\mathbf{r}_v\mathbf{n}) = 0$. Resolve the vector $\mathbf{r}_{vv}$ in terms of $\mathbf{r}_u$, $\mathbf{r}_v$, $\mathbf{n}$. We obtain

$$\mathbf{r}_{vv} = \alpha\mathbf{r}_u + \beta\mathbf{r}_v + \gamma\mathbf{n}. \qquad (*)$$

Substituting this for $\mathbf{r}_{vv}$ in $(\mathbf{r}_{vv}\mathbf{r}_v\mathbf{n}) = 0$, we get $\alpha(\mathbf{r}_u\mathbf{r}_v\mathbf{n}) = 0$, i.e., $\alpha = 0$.

Multiplying (*) throughout by $\mathbf{r}_u$, we see that $\mathbf{r}_{vv}\mathbf{r}_u = 0$. However, $\mathbf{r}_{vv}\mathbf{r}_u = (\mathbf{r}_u\mathbf{r}_v)_v - \frac{1}{2}(\mathbf{r}_v^2)_u = -\frac{1}{2}G_u$. Therefore, $G_u = 0$ along $\bar{\gamma}$, when $u = 0$.

The Gaussian curvature of a surface with the linear element $du^2 + G\,dv^2$ is known to be given by

$$K = -\frac{(\sqrt{G})_{uu}}{\sqrt{G}}.$$

Hence, for a surface of constant Gaussian curvature K, G satisfies the differential equation

$$(\sqrt{G})_{uu} + K\sqrt{G} = 0. \qquad (**)$$

Consider the following three cases, viz.,

$$(1)\ K > 0,\ (2)\ K < 0,\ (3)\ K = 0.$$

In the first, the general form of $\sqrt{G}$ satisfying (**) is

$$\sqrt{G} = A(v)\cos\sqrt{K}u + B(v)\sin\sqrt{K}u.$$

Since $G(0, v) = 1$ and $G_u(0, v) = 0$, we have $A(v) = 1$ and $B(v) = 0$. Thus, if $K > 0$, then there exists a parametrization of the surface, for which the first fundamental form is

$$ds^2 = du^2 + \cos^2\sqrt{K}u\,dv^2.$$

Similarly, in the second case,

$$ds^2 = du^2 + \cosh^2 \sqrt{-K}u\, dv^2.$$

Finally, in the third,

$$ds^2 = du^2 + dv^2.$$

Hence, *surfaces of the same constant Gaussian curvature are locally isometric.* In particular, surfaces of constant positive Gaussian curvature K are locally isometric to a sphere of radius $1/\sqrt{K}$, those of zero Gaussian curvature are locally isometric to the plane, and those of constant negative curvature to the pseudosphere.

5. Gauss-Bonnet Theorem

Consider a curve γ and its point P. The curvature of its projection onto the tangent plane at P is said to be *geodesic*. For the geodesic curvature of $\mathbf{r} = \mathbf{r}(t)$, we obtain

$$\varkappa = \frac{1}{|\mathbf{r}'|^3} (\mathbf{r}''\mathbf{r}'\mathbf{n}).$$

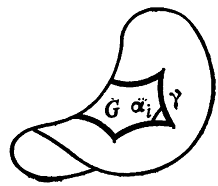

Fig. 103

We can see that *the geodesic curvature of a geodesic is zero.* It turns out that geodesic curvature is also an object of the surface intrinsic geometry.

The following *Gauss-Bonnet theorem* is valid.

Let G be a domain on a surface, homeomorphic to a circle, and bounded by a regular curve γ. Then

$$\int_\gamma \varkappa\, ds = 2\pi - \iint_G K\, d\sigma.$$

Here, integration with respect to arc length s of γ is meant on the left-hand side, and with respect to the area of G on the right, geodesic curvature $\varkappa$ assumed positive, where γ is convex outwards, and negative, where it is convex inwards.

If γ is piecewise smooth with interior angles at the break points α_i, then

$$\int_\gamma \varkappa\, ds + \sum_i (\pi - \alpha_i) = 2\pi - \iint_G K\, d\sigma,$$

with no smoothness violations in integrating along γ (Fig. 103).

In the case of a geodetic triangle (where the sides are geodesics),

$$\pi - \alpha_1 - \alpha_2 - \alpha_3 = -\iint_G K\, d\sigma.$$

In particular,

$$\alpha_1+\alpha_2+\alpha_3-\pi=\frac{\sigma}{R^2}$$

for a spherical triangle, where R is the radius of the sphere, and σ the triangle area.

6. Closed Surfaces

A simple surface is said to be *closed* if it is finite and without boundary.

Let F be a simple closed surface. Partition it into polygonal domains g_k homeomorphic to a circle, so that any two domains of the decomposition either have no common points, or have a common vertex, or a common side. Applying the Gauss-Bonnet formula to each g_k, we obtain

$$\int \varkappa^k\,ds+\sum_i(\pi-\alpha_i^k)=2\pi-\iint\limits_{g_k} K\,d\sigma,$$

and see that

$$2\pi f_2-\iint\limits_F K\,d\sigma$$

on the right-hand side if we add all these equalities together termwise, where f_2 is the number of g_k. The first addends on the left are eliminated, since a side of g_k is that of another $g_{k'}$, and $\varkappa^k=-\varkappa^{k'}$. Summing up the angles α_i^k for all i and k, we obtain the angle-sum for all the domains, which can be odne simply if we first find that of the angles with a common vertex (equal to 2π). Therefore, the sum of all α_i^k is $2\pi f_0$, f_0 being the number of vertices in the surface partition into polygonal domains.

There are as many addends π in

$$\sum_i(\pi-\alpha_i^k)$$

as there are vertices in the polygonal domain g_k, or, which is equivalent, as there are sides in it. Hence, on adding these sums together, π is counted as many times as there are sides in the decomposition of F into g_k, and then taken twice, since each belongs to two domains of the partition. Thus, the result can be represented as

$$2\pi f_1-2\pi f_0=2\pi f_2-\iint\limits_F K\,d\sigma,$$

and

$$\frac{1}{2\pi}\iint\limits_F K\,d\sigma=f_2-f_1+f_0. \qquad (*)$$

The integer

$$\chi(F) = f_2 - f_1 + f_0 \qquad (**)$$

is called *the Euler characteristic* of the surface. It follows from (**) that the *Euler characteristic does not depend on the partition of a surface into polygonal domains.*

Defined according to (**), the concept of Euler characteristic makes sense for any simple surface, not necessarily regular. It can be proved that, *in the general case, too, it does not depend on the method for partitioning the surface.* Since, under a topological transformation of a surface F into a surface F', the partition of the former into

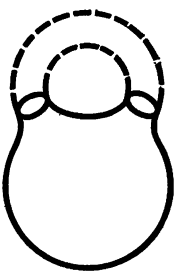

Fig. 104

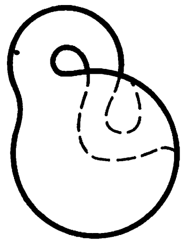

Fig. 105

polygonal domains is carried into that of the latter, f_2, f_1, f_0 remaining unchanged, *the Euler characteristic is unaltered under a topological transformation of a surface.*

We now find the Euler characteristic of a convex polyhedron (meaning its total area). Any convex polyhedron can be obtained by a topological transformation of a sphere, for which it suffices to project the latter with centre inside the former onto its surface. Hence, the Euler characteristic of a convex polyhedron is two, and *if the number of vertices of a convex polyhedron is α_0, that of edges α_1, and that of faces α_2, then*

$$\alpha_0 - \alpha_1 + \alpha_2 = 2$$

(*Euler theorem*).

Since a topological transformation does not alter the Euler characteristic, the question naturally arises as to how well a simple surface can be specified by it. It turns out that *simple surfaces with the same Euler characteristic are topologically transformable into each other.*

We illustrate by examples of different topological types of simple surfaces. Imagine an elastic sphere with two circular holes. Pull at their edges, and join them as shown in Fig. 104. The closed surface obtained is called a *sphere with a handle.* A sphere with two or more handles can be obtained similarly. The Euler characteristic of a sphere with p handles is $2 - 2p$. In particular, that of the torus is zero. It turns out that any simple closed surface can be obtained by a topological transformation of a sphere with handles.

A handle can be attached to a sphere differently, viz., by pulling the edge of one opening outwards, drawing it inside, and then joining it to the edge of another hole as in Fig. 105. The obtained figure cannot be regarded as a surface in the sense of our definition, since, as it turns out, there is no simple surface from which the figure could be locally obtained by a topological transformation. However, such surfaces do exist in four-dimensional space. Therefore, taking a somewhat generalized notion of surface, we can also regard such figures as general surfaces.

We speak of the surface constructed that it is obtained by attaching to the sphere a *handle of the second kind*, and is unilateral, i.e., we can go from inside the sphere outside, and vice versa. Unilateral surfaces are also said to be *non-orientable*.

EXERCISES TO CHAPTER XII

1. Given the linear element $ds^2 = \lambda\,(du^2 + dv^2)$ of a surface, show that its Gaussian curvature is

$$K = -\frac{1}{2\lambda}\left(\frac{\partial^2 \ln \lambda}{\partial u^2} + \frac{\partial^2 \ln \lambda}{\partial v^2}\right)$$

2. Given the linear element $ds^2 = du^2 + 2\cos\omega\, du\, dv + dv^2$ of a surface, show that its Gaussian curvature is

$$K = -\frac{\omega_{uv}}{\sin\omega}.$$

3. Prove that if coordinate lines are those of curvature, then the Peterson-Codazzi equations are

$$L_v = HE_v, \quad N_u = HG_u.$$

4. Prove that a surface of zero mean curvature can be parametrized so that its first and second fundamental forms are

$$\mathrm{I} = \lambda\,(du^2 + dv^2),$$
$$\mathrm{II} = du^2 - dv^2.$$

5. Show that an asymptotic geodesic line is straight.

6. Show that if a geodesic line is that of curvature, then it lies in a plane.

7. Prove that cylindrical surface geodesics meet the rectilinear generators at the same angle.

8. Find the geodesic lines on a surface with the linear element $ds^2 = \frac{du^2+dv^2}{v^2}$.

9. Prove that cylindrical and conical surfaces as well as those formed by the tangents to space curves are locally isometric to a plane.

10. A sphere of unit radius is described from the vertex of a convex polyhedral angle. Find the area of the contained sphere if the sum of the dihedral angles is α.

11. Prove that the sum of the angles of a geodetic triangle on a surface of positive Gaussian curvature is greater than π, and less than π on a surface of negative curvature.

12. Prove that the area of any geodetic triangle is not greater than π/a^2 if it lies on a surface of negative Gaussian curvature $K \leqslant -a^2$.

13. Find the Euler characteristic of a torus.

14. What is the Euler characteristic of a closed surface, given that it is topologically equivalent to a sphere with n handles of the first kind?

Part Three

FOUNDATIONS OF GEOMETRY

Chapter XIII

HISTORICAL SURVEY

1. Euclid's *Elements*

Geometry began as an empirical science, and became especially much developed with the Egyptians who applied it to earth measurement and irrigation work.

In the first millennium B.C., the Egyptians' geometric knowledge was adopted by the Greeks, thus starting a new stage. The Greek geometers of the 7th-3rd cc.B.C. not only enriched the science with new facts, but also took important steps towards the formulation of a rigorous logical sequence.

The many-century work was summarized and systematized by Euclid (330-275 B.C.) in his famous *Elements*. Euclid for the first time introduced a strictly logical account of geometry. Its treatment was so immaculate for his epoch that, during two thousand years since the *Elements* appeared, the book has remained a unique geometry manual. Books I-IV and VI of the whole number of thirteen were devoted to geometry proper, and accounted for its plane chapters, as well as Books XI-XIII embracing solid geometry. The others contained arithmetic in geometric treatment. Each opened with a definition of new concepts. E.g., Book 1 contained 23 definitions.

In particular,

DEFINITION 1. *A point is that which has no part.*

DEFINITION 2. *A line is breadthless length.*

DEFINITION 3. *A straight line is a line which lies evenly with the points on itself.*

The definitions were followed by postulates and axioms (common notions).

E.g.,

Postulate 1. It is postulated to draw a straight line from any point to any point.

Postulate 5. It is postulated that, if a straight line falling on two straight lines makes interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Axiom 1. Things which are equal to the same thing are also equal to one another.

Axiom 2. If equals be added to equals, the wholes are equal.

Both postulates and axioms were assumed without proof, it remaining unknown by which principle some statements were taken as postulates, and others as axioms.

Axioms were followed in a strict sequence by theorems and construction problems under the general title *Propositions*, so that the proof or solution of each subsequent statement was based on the previous. Here is one of them.

If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.

Though the *Elements* have been a paragon for a very long time, they did not at all attain the modern level of rigor. The definitions of geometric objects in the first book were given in a manner of descriptions, and not at all perfect at that. E.g., Definition 4 of a straight line did not make it different from a circumference, whereas Definition 2 of an arbitrary line mentioned length and breadth which should have been defined themselves.

We must not think, however, that all the definitions preceding the first book were defective. On the contrary, part of those, including a circumference, triangle, right, acute and obtuse angles, were either flawless or insignificantly imprecise, which can be easily rectified. Meanwhile, if we remember that the properties of geometric objects, described by the inaccurate definitions were never used in proof, then they can be omitted without any detriment to the account.

As to the postulates and axioms, their formulations were irreproachable, the statements essential, and formed the basis for the subsequent proofs.

Finally, we turn just to them. The proofs of all the propositions, as conceived by the author of the *Elements*, had to be eventually based on the geometric object properties determined by the postulates and axioms. However, even cursory familiarity with Euclid's proofs shows that a number of such properties and relations among geometric objects could not be clarified either by postulates or axioms. E.g., in the proof for the above-mentioned proposition on the congruence of triangles, Euclid made use of a motion, and referred in some others to the properties of mutual disposition of points on a straight line, expressed by the term "between".

The question naturally arises if we can free the Euclidean proofs of this defect by possibly replacing them with others based only on postulates and axioms. The answer has been obtained comparatively not long ago. It turned out that this could be done only by a suitable completion of the Euclidean postulates and axioms.

2. Attempts to Prove the Fifth Postulate

Certain of the above defects of the *Elements* have already been noticed by the ancient Greeks, due to which attempts to improve the treatment were made. The principal goal was to reduce the Euclidean postulate and axiom system to minimum.

The natural way to solve the problem is in deducing some of the postulates and axioms from the others. The *Elements* were just in this way stripped of the fourth postulate (where the equality of all right angles was meant).

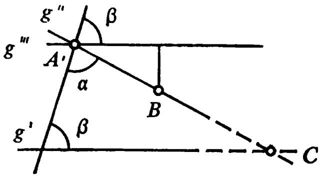

Fig. 106

However, all the efforts to get rid of the fifth postulate were of no avail, though geometers have tried to do it for more than two thousand years. The typical mistake of most of its proofs was either purposeful or accidental use of some or other statement not explicitly contained in the remaining postulates and axioms, and not following from them.

E.g., here is the proof of Proclus.

Given $\alpha + \beta < 2d$ (Fig. 106), *prove that the straight lines* g' *and* g'' *meet at a certain point* C.

Draw through the point A a straight line g''' parallel to g'. Take a point B on g'', and drop the perpendicular to g''' from it. Since the distance from g''' increases without limit as that between B and A grows, and the distance between g' and g''' is constant, there is a point C on g'' belonging to g'. This is just where g' and g'' meet.

The property of parallel straight lines, to which we have resorted in the proof, is not explicitly contained in the other postulates or axioms. Moreover, it cannot be deduced from them.

The fifth postulate can be proved on the basis of a great many other statements.

E.g.,

(i) *All perpendiculars to one side of an acute angle cut its other side.*

(ii) *There exist similar triangles which are not congruent.*

(iii) *There exist triangles of arbitrarily large area.*

(iv) *There exist triangles whose angle-sum is equal to two right angles.*

(v) *Through a point outside a given straight line, not more than one parallel line can be drawn.*

Though the attempts to prove the fifth postulate did not lead to the desired result, they undoubtedly played a positive part in the development of geometry, often enriching it with new interesting theorems whose proofs were not based on the fifth postulate. One of them proved by A. Legendre states that *the sum of the angles of any triangle is not greater than two right angles.*

3. Discovery of Non-Euclidean Geometry

One of the methods to which many geometers of the 18th c. and the first half of the 19th c. resorted in hope to prove the fifth postulate consisted in replacing it by its negation or some other statement equivalent to the negation. All possible propositions logically following from the postulate and axiom system so altered were then proved similarly to the method used in the *Elements*. If the fifth postulate does, in fact, follow from the other postulates and axioms, then the postulate and axiom system so formed is self-contradictory. Therefore, we shall sooner or later come to two mutually exclusive results, thus proving the fifth postulate.

G. Saccheri, J. Lambert and A. Legendre tried to prove it exactly in this manner.

The first one considered a rectangle with two right base angles and equal non-adjacent sides (Fig. 107). There can be three hypotheses

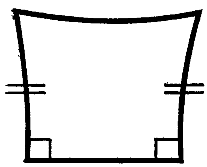

Fig. 107

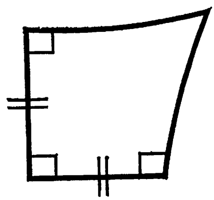

Fig. 108

regarding the other two angles which are obviously equal, viz., that they are right, obtuse or acute. He proved that the right-angle hypothesis and fifth postulate were equivalent, i.e., the latter could be proved on postulating the former and vice versa. Having postulated the obtuse-angle hypothesis, G. Saccheri came to a contradiction, and, finally, postulated that of acute angles, deriving various corollaries which are absurd from the point of view of customary geometric ideas. E.g., *parallel lines either possess only one common perpendicular, on both sides of which they diverge without limit, or have none, and, approaching each other asymptotically in one direction, diverge in the other without limit.*

G. Saccheri made no conclusion that a contradiction was obtained only because of the derived results being contrary to the usual ideas about straight line disposition, and was stubbornly looking for a logical absurdity. Such a contradiction was eventually "found" by him—however, due to a computational error.

A similar construction was considered by J. Lambert who took a quadrilateral with three right angles (Fig. 108), and, similarly to G. Saccheri, investigated the three hypotheses for the angle at the fourth vertex. Proving that the right-angle hypothesis is equivalent to the fifth postulate, that of an obtuse angle is impossible, and, having postulated the acute-angle conjecture similarly to G. Saccheri, he obtained numerous corollaries which reveal paradoxical properties of straight line disposition.

Nevertheless, as well as G. Saccheri, J. Lambert did not see any contradiction. He could not find any logical contradiction either; still, the acute-angle hypothesis was not rejected.

Developing corollaries to the acute-angle hypothesis, J. Lambert discovered that they were analogous to the geometry on the sphere, and expressed the correct conjecture for this hypothesis "to be valid on some imaginary sphere". Among the 17th c. geometers, it was J. Lambert who stood nearest to the correct solution of the fifth-postulate problem.

In his "proof" of the fifth postulate, A. Legendre considered the following three hypotheses regarding the angle-sum of a triangle, viz.,

(i) The sum of the angles of a triangle is equal to two right angles.

(ii) The sum of the angles of a triangle is greater than two right angles.

(iii) The sum of the angles of a triangle is less than two right angles.

He proved that the first hypothesis is equivalent to the fifth postulate, and that the second one is impossible. Finally, accepting the third hypothesis, he also came to a contradiction by implicitly making use of the fifth postulate through one of its equivalents.

The great Russian mathematician N.I. Lobachevsky (1792-1856) who is honoured for the discovery of a new geometry, *Lobachevskian geometry*, also began with an attempt to prove the fifth postulate.

As is shown above (Sec. 2), one of the fifth postulate equivalents is in the statement that not more than one straight line parallel to a given one passes through an outside point. N.I. Lobachevsky replaced the fifth postulate by the following.

At least two straight lines not intersecting a given one pass through an outside point.

Similarly to his predecessors, N.I. Lobachevsky hoped to find a contradiction in the Euclidean corollary system so altered. However, having developed his theory to make it on par with the *Elements* in contents, N.I. Lobachevsky saw that the system was non-contradictory, and draw a remarkable conclusion regarding the existence of a geometry different from Euclidean, with the fifth postulate not holding. It all happened in 1826.

At first glance, N.I. Lobachevsky's conclusion may seem insufficiently well-founded. In fact, how can it be guaranteed that there would be no contradiction if we developed his theory further? Nevertheless, the same can be also applied to Euclidean geometry, so that, from the standpoint of logical consistency, both geometries are equivalent. Moreover, subsequent investigations have shown that they are closely related, with the logical consistency of one depending on that of the other.

Thus, both Euclidean and Lobachevskian geometries are equivalent as logical systems. Which of them reflects space relations in the surrounding world better can be found only by experience. N.I. Lobachevsky understood this himself, and measured the angle-sum of an astronomical triangle to the purpose.

N.I. Lobachevsky was the first, but not the only geometer, who discovered the existence of a geometry different from Euclidean. The new geometry was also discovered by F. Gauss who wrote about it in his letters.

Three years after N.I. Lobachevsky's work had seen the light, the Hungarian mathematician J. Bolyai (1822-1860), being unaware of his predecessor's research, published a paper with an account of the same theory, but in a less developed form.

4. Works on the Foundations of Geometry in the Second Half of the 19th Century

Not many of N.I. Lobachevsky's contemporaries understood him, and agreed with his discovery. The majority, among whom there were many great mathematicians, treated it sceptically.

The universal recognition of Lobachevskian geometry was considerably assisted by the after-Lobachevsky geometers, and, first of all, by E. Beltrami (1862) who proved that the Lobachevsky plane geometry is valid on a surface of constant negative curvature if hyperbolic lines are thought of as geodesics, while a motion is understood in the sense of isometric mapping of the surface onto itself.

This was a proof that Lobachevskian geometry is non-contradictory. Indeed, a contradiction in it would correspond in the above interpretation to that in the theory of Euclidean surfaces, i.e., to one in Euclidean geometry.

A vulnerable point in the proof of Lobachevskian geometry consistency, if it is based on the Beltrami interpretation is, as D. Hilbert had demonstrated, that there exists no complete Euclidean surface of constant negative curvature without singularities, and, therefore, the geometry of only part of the Lobachevsky plane can be interpreted on it. The drawback was eliminated in the later models by H. Poincaré and F. Klein.

F. Klein interpreted hyperbolic plane geometry inside a circle in the Euclidean plane, where its chords are understood as straight lines, and motions as collineations preserving the circumference. The proof on the basis of this interpretation that Lobachevskian geometry is consistent will be seen to be irreproachable. We reproduce it in Ch. XV.

At the same time, it is the substantiation of the fifth postulate independence from the other Euclidean postulates and axioms. In fact, if the fifth postulate were a corollary to the other postulates or axioms, then Lobachevskian geometry would be contradictory as containing two mutually exclusive statements, the Lobachevskian and Euclidean fifth postulates.

The general tendency to mathematical rigor, which marked all works in the second half of the 19th c., and the solution to the fifth postulate problem made geometers subject the geometric axiom system to a thorough investigation. The researches showed that the Euclidean axioms are not at all perfect, first of all because they are incomplete. As we shall see later, they omit a number of axiom groups absolutely necessary for strict proofs, and the Euclidean axiom system was then completed with lacking axioms. Thus, M. Pasch (1882) supplied the Euclidean axiomatics with the axioms of order. One of them now bears his name.

The Euclidean axiom study was completed by D. Hilbert in 1899. The axiom system given by D. Hilbert consists of five groups, viz., axioms of incidence, axioms of order, congruence axioms, axioms of continuity and the parallel axiom, all referring to objects of three kinds, i.e., points, straight lines, planes, and the three relations among them, expressed by the terms "incident", "between" and "congruent". What is a point, straight line or plane, and what is the true meaning of the above relations, was not made precise. Everything assumed as known is expressed by the axioms, and the geometry constructed thus admits concrete realizations which can go very far from the usual ideas.

D. Hilbert subjected this system of axioms to a very profound and comprehensive investigation. In particular, he proved that it is non-contradictory if arithmetic is non-contradictory. Further, besides that of parallelism, he showed the independence of certain other axioms, and, finally, investigated the problem of how far a geometry can be developed if some or other axiom groups into which the whole system is divided are taken as its basis.

D. Hilbert almost completed the many-century work on the foundations of elementary geometry. It was very highly assessed by the contemporaries, and awarded the Lobachevsky prize in 1903.

5. System of Axioms for Euclidean Geometry according to D. Hilbert

The system of axioms for Euclidean geometry according to D. Hilbert consists of five groups, viz., axioms of incidence, axioms of order, axioms of congruence, the parallel axiom and axioms of continuity.

Axioms of incidence determine the properties of mutual disposition of points, straight lines and planes, expressed by the term "incident" or some equivalent ones.

I_1. For any two points A and B, there is a straight line incident with each of these points.

I_2. For any two points A and B, there exists not more than one straight line incident with each.

I_3. There exist at least two points in a straight line. There exist at least three points not in the same straight line.

I_4. For any three points A, B and C not in the same straight line, there exists a plane incident with each. For any plane, there always exists a point incident with it.

I_5. For any three points A, B, C not in the same straight line, there exists not more than one plane incident with these points.

I_6. If two points A and B of a straight line a are in a plane α, then every point of a is in that plane.

I_7. If a point A is in two planes α and β, then there exists at least one other point B in α and β.

I_8. There exist at least four points not in a plane.

Axioms of order express the properties of mutual disposition of points in a straight line or plane, determining the concept "between".

II_1. If a point B is between points A and C, then A, B and C are distinct points, and B is also between C and A.

II_2. For any two points A and C, there exists at least one point B in the straight line AC, so that C is between A and B.

II_3. Of any three points in a straight line, not more than one is between the other two.

The term "between" for points in a straight line permits us to define the concept of line segment in the usual manner.

II_4. Let A, B and C be three non-collinear points, and a a line in the plane ABC, which does not contain A, B or C. Then if a contains a point of the segment AB, a will also contain a point of the segment AC or a point of the segment BC (*Pasch axiom*).

The axioms of congruence determine the concept of "congruence", or equality, for line segments and angles.

III_1. If A and B are two distinct points in a straight line a, and A' a point in the same or another line a', then there exists a point B' on the same side of a' as A', so that the line segment AB is congruent to the line segment $A'B'$.

III_2. If two line segments are congruent to a third, then they are congruent to each other.

III_3. Let AB and BC be two segments on a line a such that AB and BC share only the point B in common. Furthermore, let $A'B'$ and $B'C'$ be segments on line a' such that $A'B'$ and $B'C'$ share only B' in common. Then if AB is congruent to $A'B'$ and BC is congruent to $B'C'$, we have AC congruent to $A'C'$.

An *angle* is defined as a figure consisting of two different rays emanating from the same point.

III_4. One, and only one, angle congruent to a given angle can be marked off a given half line into a given half-plane determined by this half line and its extension.

III_5. If $AB \equiv A_1B_1$, $AC \equiv A_1C_1$ and $\angle A \equiv \angle A_1$ in two triangles ABC and $A_1B_1C_1$, then $\angle B \equiv \angle B_1$, $\angle C \equiv \angle C_1$.

IV. *Parallel axiom.* Let a be an arbitrary straight line, and A an outside point; then there exists not more than one straight line through A, not intersecting a in the plane determined by a and A.

The *axioms of continuity.*

V_1 (*Archimedes' axiom*). Let AB and CD be two line segments. Then there exist a finite number of points $A_1, A_2, \ldots, A_n$ in the straight line AB, so that the line segments $AA_1, A_1A_2, \ldots, A_{n-1}A_n$ are congruent to CD, and the point B is between A and A_n.

V_2 (*Axiom of linear completeness*). The set of points in a straight line, satisfying axioms of order, the first axiom of congruence and Archimedes' axiom, does not admit any extension, i.e., no points can be added to this set, so that all the axioms hold.

Chapter XIV

SYSTEM OF AXIOMS FOR EUCLIDEAN GEOMETRY AND THEIR IMMEDIATE COROLLARIES

1. Basic Concepts

It is rather complicated to give Euclidean geometry deductive structure on the basis of the Euclid-Hilbert axioms. Difficulties arise almost at once, in introducing the concept of measure of line segments and angles; accordingly, we resort to another axiom system where these problems are eliminated.

In our treatment, the *basic concepts* are a point, straight line and plane, the relation of incidence for points, straight lines and planes, expressed by the term "incident", that of order for points in a straight line, expressed by the terms "between", "length" for line segments and "measure of angles in degrees". These concepts are not defined, and everything assumed known about them is given axiomatically.

The axiom system which we shall employ mostly coincides with the axiomatics of the school geometry course; however, it is somewhat weakened. In particular, the axiom of marking off a line segment of given length on a half-line from its origin is replaced by the weaker axiom of existence of a line segment of given length, and the axiom of constructing an angle is omitted at all. Their introduction at school is due to purely methodological reasons, and aimed at the simplicity of the treatment at the beginning of the course.

For convenience, we first formulate axioms for the plane, and then introduce the group of axioms C for space. The axioms for the plane are naturally divided into groups in accordance with the basic concepts of incidence, order and measure.

2. Axioms of Incidence

The axioms of incidence determine the properties of mutual disposition of points and straight lines, given by the term "incident". Meanwhile, the expression "a point is incident with a straight line", "a point lies in a straight line" and "a straight line passes through a point" are assumed to be equivalent.

If a point is incident with two straight lines, then we will say that they *intersect* at the point, or that it is the point of their intersection.

The group of the axioms of incidence includes the following two.

Axiom I_1. For any two points, there exists one, and only one, straight line passing through them.

Axiom I_2. In each straight line, there are at least two points. There exist three points not in the same straight line.

It follows from Axiom I_1 that *two straight lines either do not intersect or intersect only at one point.* In fact, if they had at least two intersection points, the straight lines would pass through these points, which is contrary to Axiom I_1. According to the latter, only one straight line passes through any two points. It follows that a straight line is completely determined by specifying two of its points, thus making it possible to denote a straight line by two points (e.g., a straight line AB).

It follows from Axiom I_2 that, *for any straight line, there exists a point not in this line.* Indeed, of the three points whose existence is stated by Axiom I_2, at least one is outside the given straight line.

The axiom corresponding to Axiom I_2, and given in the school treatment of the subject, requires that there should be points (therefore, at least two) in each straight line, and that there should be points outside it. In this form, the axiom is taken by the student as something that goes without saying. The statement in the form of I_2 with two points in a straight line may cause confusion, since the visual image of a straight line assumes the existence of an infinite set of points in it and outside.

3. Axioms of Order

The axioms of order express the properties of mutual disposition of points in straight lines and planes. Meanwhile, the relation of mutual disposition of points in a straight line is used and expressed by the term "between".

Axiom II_1. Of any three points in a straight line, one, and only one, is between the other two.

The expression "a point B is between points A and C" is equivalent to "a point B separates the points A and C", or "the points A and C are on opposite sides of the point B". If B separates A and C, then according to Axiom II_1, A does not separate B and C. Instead, B and C can be said to *lie* on the same side of A.

The concept of straight line segment is introduced by means of that of "between" for points in a straight line. Viz., part of the straight line between two points A and B, i.e., the set of its points between A and B, is called the *line segment* AB.

Axiom II_2. A straight line separates the set of points in a plane, which are not incident with it, into two subsets (half-planes), so that the line segment joining two points in one half-plane does not meet the straight line, whereas the line segment joining two points in different half-planes does meet it.

We call part of a straight line AB consisting of all those points which are on the same side of the point A along with the point B, the *half-line*, or *ray*, AB. A is called the *origin* of the half-line.

Draw through the origin A of the half-line AB any straight line a different from the straight line AB. Then *the half-line AB consists of those, and only those, points of the straight line AB, which are in the same half-plane as the point B with respect to a.* In fact, for any straight line a, any line segment of the straight line AB can intersect a only at A. It follows that if X is a point in the half-line AB, then the segment BX does not intersect a, i.e., X and B are in the same half-plane. If X is a point in the straight line AB in the same half-plane with B, then the segment BX does not intersect a; therefore, X and B are on the same side of A, i.e., X belongs to the half-line AB, and the statement is thus proved.

A point A in a straight line a divides this straight line into two half-lines, and is the origin of each. The points in one half-line are not separated by A, whereas those in different half-lines are separated by it. For proof, it suffices to draw through A a straight line b different from a. Then parts of a in different half-planes with respect to b are just the half-lines in question. The half-lines of one straight line with the common origin are said to be *complementary*.

A half-line is completely determined by specifying its origin and some other point, which justifies the notation of a half-line by two points (e.g., a half-line AB), the origin placed first.

A *triangle* is a figure consisting of three points not in one straight line, and three line segments joining the points pairwise. The points are called the *vertices* of the triangle, and the line segments joining them its *sides*.

It follows from Axiom II_2 that *if a straight line not passing through any vertex of a triangle intersects one of its sides, then it intersects one, and only one, of the other two sides.*

In fact, let ABC be a triangle, and a a straight line intersecting its side AB. The points A and B are in different half-planes with respect to a. The point C is in one of them. If C is in the same half-plane with A, then a does not intersect the line segment AC, but does intersect the line segment BC. If C is in the same half-plane with B, then a does not intersect BC, but does intersect AC. In both cases, a intersects one, and only one, of the sides AC or BC of the triangle. This theorem is taken in the Hilbert axiomatics as an axiom, and called the *Pasch axiom.*

4. Axioms of Measure for Line Segments and Angles

Axiom III_1. Each line segment is of length greater than zero. If a point C is in a line segment AB, then its length is equal to the sum of those of the line segments AC and BC.

Introducing this axiom into the school course, we rely on the student's understanding how a line segment is measured by means of some known tool, e.g., a ruler with scale marks. However, it should be borne in mind that Axiom III_1 does not at all assume any measurement. It only states the possibility to associate any line segment with a number (its length), so that the conditions of the axiom are fulfilled.

On the other hand, we should not think that the length of a line segment whose existence is stated by Axiom III_1 is something different from what we obtain by making measurements in the usual way. Nevertheless, this requires proof (see Ch. XVIII, Sec. 1).

Axiom III_1 permits us to introduce coordinates on a straight line, i.e., associate each point in the line with a real number, so that if $x(A)$ and $x(B)$ are the coordinates of two points A and B, then the length of the line segment AB equals $|x(B) - x(A)|$.

In fact, let O be a point in the straight line. We associate it with zero as its coordinate. O divides the straight line into two half-lines. We agree to call one of them the *positive side* of O, and the other the *negative side* of O. Now, if a point A is on the positive side, then its coordinate $x(A)$ is the length of the line segment OA; if A is on the negative side, then its coordinate is a negative number whose absolute value is the length of OA.

We show that the length of the line segment AB equals $|x(B) - x(A)|$. If the points A and B are in different half-lines, then the length of AB equals the sum of those of OA and OB. Therefore, $AB = |x(B) - x(A)|$. Assume that A and B are on the same side, e.g., positive. Of the three points O, A and B, one lies between the other two. It cannot be O, since A and B are in one half-line. Hence, this is either A or B, e.g., B. Then the length of OA equals the sum of those of OB and BA. Therefore, the length of AB equals $x(A) - x(B) = |x(B) - x(A)|$. The other cases of mutual disposition of O, A and B are considered similarly.

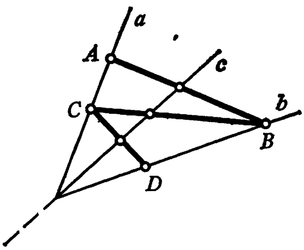

Fig. 109

An *angle* is a figure formed by two different half-lines called its *sides*, with a common origin called its *vertex*. If the sides of an angle are complementary half-lines of one straight line, then the angle is said to be *straight*.

We will say that a ray c is *between* the sides of an angle (ab) if it emanates from its vertex, and intersects some line segment with the end-points on the angle sides. In the case of a straight angle, we assume that *any ray emanating from its vertex, and different from its sides, is between the sides of the angle.*

It is easy to see that *if a ray is between the sides of an angle, then it intersects any line segment with the ends on the sides of the angle* (Fig. 109). In fact, by definition, a ray c intersects some line segment AB whose ends are on the angle sides. Let CD be another such line segment. Applying the Pasch theorem to the triangle ABC, straight line containing c, triangle BCD and the straight line again, we conclude consecutively that C meets BC and CD.

Axiom III$_2$. Each angle has a certain measure in degrees greater than zero. A straight angle has 180°. If a ray c is between the sides of an angle (ab), then the measure in degrees equals the sum of those of the angles (ac) and (bc).

We note the following theorem.

If we mark off on a half-line a and its extension two angles (ab) *and* (ac) *lying in the same half-plane, then either the ray c is between the sides of* (ab), *or the ray b is between the sides of* (ac). *In any case,* $(bc) = |(ac) - (ab)|$.

Proof. Take a point A in the ray a, a point A_1 in its complement, and a point C in the ray c. The straight line containing the ray b intersects the side AA_1 of the triangle ACA_1; therefore, by the Pasch theorem, it intersects either the side AC or the side A_1C just with b, since the complementary ray is in the other half-plane. If b intersects the line segment AC (Fig. 110a), then it is between the sides of the angle (ac), with $(ac) = (ab) + (bc)$. Hence, $(bc) = (ac) - (ab)$.

Assume that b intersects A_1C at a point D (Fig. 110b). Applying the Pasch theorem to the triangle ADA_1 and straight line containing c, we conclude that c intersects the line segment AD, and, therefore,

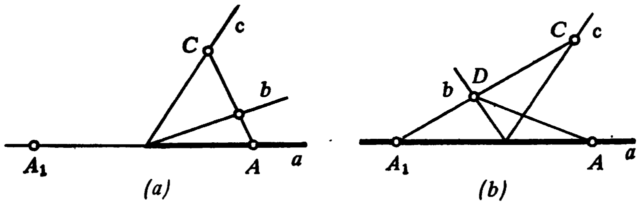

Fig. 110

is between the sides of (ab). Meanwhile, $(ab) = (ac) + (bc)$; hence, $(bc) = |(ac) - (ab)|$, and the theorem is thus proved completely.

5. Axiom of Existence of a Triangle Congruent to a Given One

Two lines are called *equal* (or *congruent*) if they are of equal length. Two angles are called *equal* (or *congruent*) if they have the same measure in degrees. Two triangles ABC and $A_1B_1C_1$ are called *congruent* if $\angle A \equiv \angle A_1$, $\angle B \equiv \angle B_1$, $\angle C \equiv \angle C_1$, $AB \equiv A_1B_1$, $BC \equiv B_1C_1$, $AC \equiv A_1C_1$. Briefly, it is expressed by saying that *two triangles are congruent if the corresponding sides and corresponding angles are equal.* The correspondence between the vertices and sides of congruent triangles is reflected in the notation of their vertices. If we say that a triangle ABC is congruent to a triangle $A_1B_1C_1$, then the corresponding vertices are A and A_1, B and B_1, C and C_1, and the corresponding sides AB and A_1B_1, AC and A_1C_1, BC and B_1C_1. To designate the congruence of triangles, the usual symbol will be used (e.g., $\Delta ABC \equiv \Delta A_1B_1C_1$). Meanwhile, it is important in what order the vertices are written. $\Delta ABC \equiv \Delta A_1B_1C_1$ means that $\angle A \equiv \angle A_1$, $\angle B \equiv \angle B_1$, . . ., whereas $\Delta ABC \equiv \Delta B_1A_1C_1$ quite a different fact, viz., $\angle A \equiv \angle B_1, \angle B \equiv \angle A_1$, ...

Axiom IV. Let ABC be a triangle, and a a half-line. Then there exists a triangle $A_1B_1C_1$ congruent to the triangle ABC, in which the vertex A_1 coincides with the origin of the ray a, the vertex B_1 is in a, and the vertex C_1 is in the given half-line with respect to the straight line containing a.

It follows that *we can mark off one, and only one, line segment equal to a given one from the origin of a given half-line.*

In fact, let a be a given half-line, and AB a given line segment. Take a point C outside the straight line AB. By Axiom IV, there exists a triangle $A_1B_1C_1$ congruent to the triangle ABC, in which

A_1 is the origin of the ray a, and the vertex B_1 is in the ray. The line segment A_1B_1 equals the line segment AB, since $\Delta ABC \equiv \Delta A_1B_1C_1$.

We now prove the uniqueness of the line segment. Assume that we can mark off two line segments OX and OY equal to a given one, and, therefore, equal to each other, on a half-line with the origin O. Of the three points O, X and Y, one lies between the other two. This cannot be O, since X and Y are not separated by the origin of the half-line. If this is X, then $OY = OX + XY$, which is impossible, since $OX \equiv OY$, and $XY > 0$. It is proved similarly that Y cannot lie between O and X; a contradiction, and the statement is thus proved.

It follows from Axiom IV that *one, and only one, angle equal to a given one can be marked off on a given half-line into the half-plane determined by this half-line and its extension.*

In fact, let ABC be the given angle. By Axiom IV, there exists a triangle $A_1B_1C_1$ congruent to the triangle ABC, in which A_1 coincides with the origin of the ray, the vertex B_1 is in the ray, and the vertex C_1 is in the given half-plane. The angle $A_1B_1C_1$ equals the angle ABC, since $\Delta ABC \equiv \Delta A_1B_1C_1$.

To prove the uniqueness, we assume that two angles (ab) and (ac) equal to the given angle can be marked off on the half-line a. We know that then $(bc) = |\,(ac) - (ab)\,| = 0$, which is contrary to the positiveness of the angle (bc) measured in degrees, and the uniqueness is thus proved.

6. Axiom of Existence of a Line Segment of Given Length

Axiom V. For any real number $d > 0$, there exists a line segment of length d.

It follows from Axiom V that *one, and only one, line segment of any prescribed length can be marked off on any half-line from its origin.*

In fact, by Axiom V, there exists some line segment AB of given length. It was shown in the previous section that one, and only one, line segment equal to AB can be marked off from the origin of a given half-line.

It also follows that *the introduction of coordinates on a straight line establishes a one-to-one correspondence between its points and real numbers.* Indeed, since a line segment of any prescribed length can be marked off on the positive and negative sides of the origin O, then a mapping of the set of points in the straight line onto the set of real numbers, under which the points in the straight line are associated with their coordinates, is one-to-one.

We now prove the following theorem:

For any real number $\theta < 180°$, *one, and only one, angle* (ab) *whose* *measure in degrees is* θ *can be marked off on a given half-line a into a given half-plane.*

Proof. First of all, we notice that there are angles whose measures in degrees may be arbitrarily small. In fact, let ABC be any angle other than straight, and α its measure in degrees. Take a point D in the line segment AC. Then $\angle ABC = \angle ABD + \angle CBD$. Therefore, the measure of at least one of $\angle ABD$ and $\angle CBD$ is in degrees not greater than $\alpha/2$. The existence of an angle whose measure in degrees is not greater than $\alpha/4$, etc., is proved similarly. Thus, there exist angles whose measure in degrees may be arbitrarily small.

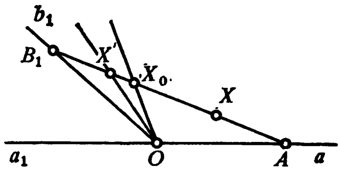

Fig. 111

Now, let a_1 be the ray complementary to a ray a (Fig. 111).

Mark off an angle (a_1b_1) less than $180° - \theta$ on a_1 into the given half-plane. By the property of supplementary angles, the angle (ab_1) is greater than θ.

Take a point A on a, and a point B_1 on the ray b_1. Let X be an arbitrary point of the segment AB_1. Denote by $M(\theta)$ the set of those points X of AB_1, for which the angle AOX is not greater than θ. Let d be the supremum of the lengths of the segments AX if $X \subset M(\theta)$, and X_0 such a point of the segment that $AX_0 = d$ (Fig. 111). We state that the angle AOX_0 equals θ.

Assume that $\angle AOX_0 = \alpha < \theta$. Mark off on the half-line OX_0 into the half-plane with the point B_1 a sufficiently small angle X_0OX' less than the angle X_0OB_1, and less than $\theta - \alpha$. Then the angle AOX' is less than θ, which is impossible, since $AX' > AX_0 = d$, and the point $X'$is incident with $M(\theta)$.

Assume now that $\angle AOX_0 = \alpha > \theta$. Mark off a sufficiently small angle X_0OX' on the half-line OX_0 into the half-plane with A, less than the angle X_0OA, and less than $\alpha - \theta$. Then the angle AOX' is greater than θ. By definition of X_0, there exist points X'' arbitrarily near to it so that the angle AOX'' is not greater than θ. The point X' is in the line segment $X''A$. Therefore, the angle AOX' is less than θ; a contradiction. Thus, the angle AOX_0 is equal to θ. Its uniqueness has been proved earlier.

The complexity of the above proof, and it can hardly be made essentially simpler, accounts for the fact that, in the school treatment, this statement is taken as an axiom.

Accordingly, the question naturally arises, can the axiom of marking off on a half-line a line segment of given length be omitted in the school axiomatics, too, and not replaced by a weaker axiom of the existence of a line segment of given length? It turns out that this cannot be done (see the proof in Ch. XV).

7. Parallel Axiom

Two straight lines on a plane are said to be *parallel* if they do not meet.

Axiom VI. Through a point not in a given straight line, not more than one straight line parallel to the given passes in the plane.

It follows that *the property of parallelism of straight lines is transitive.* Viz., if a straight line a is parallel to a straight line b, and b is parallel to a straight line c, then a is parallel to c. In fact, if a and c met, then two straight lines parallel to b, viz., a and c, would pass through the point of their intersection, which is contrary to Axiom VI.

It also follows from Axiom VI, that *if a straight line intersects one of two parallel straight lines, then it also intersects the other.* In fact, let a straight line c intersect one of two parallel straight lines a and b, say, b, but not intersect the other, i.e., a. Then two straight lines parallel to a, viz., b and c, pass through the point where b and c meet, which is contrary to Axiom VI.

8. Axioms for Space

Axiom C_1. For any plane, there exist points incident with it, and points not incident with it.

Axiom C_2. If two distinct planes have a point in common, then they intersect in a straight line.

Axiom C_3. If two distinct straight lines have a point in common, then there is one, and only one, plane through them.

Note several corollaries to the axioms for space.

There is one, and only one, plane through a straight line and an outside point.

Proof. Let a be the given straight line, and B a point not in it (Fig. 112). Take a point A in a. Such a point exists by Axiom I_2. Draw a straight line b through A and B (Axiom I_1). The lines a and b are different, since B of b does not lie in a. The lines a and b also have a common point, A. Draw a plane α through a and b (Axiom C_3). It passes through a and B.

We now show that, passing through a and B, α is unique. Assume that there exists another plane α' passing through a and B, and different from α. By Axiom C_2, α and α' intersect in a straight line. Therefore, any three points common to α and α' are in a straight line. But B and any two points in a are sure not to be in one straight line. The contradiction completely proves the theorem.

If two points in a straight line lie in a plane, then the whole line lies in the plane.

Proof. Let a be a given straight line, and α a given plane (Fig. 113). By Axiom I_2, there exists a point A not in a. Draw a plane α' through

a and A. If α' coincides with α, then α contains a, which is just what is stated by the theorem. If α' is different from α, then they intersect in a straight line a' containing two points of a. By Axiom I_1,

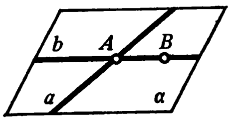

Fig. 112

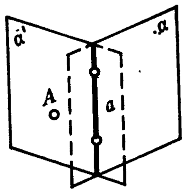

Fig. 113

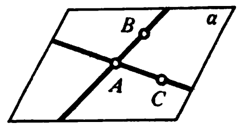

Fig. 114

a' coincides with a; therefore, a is in α, thus completing the proof.

One, and only one, plane can be drawn through three points not in the same straight line.

Proof. Let A, B, C be the three given points not in the same straight line (Fig. 114). Draw the straight lines AB and AC. They are different, since A, B and C are not in the same straight line. By Axiom C_3, a plane containing A, B, C can be drawn through AB and AC.

Prove that the plane α passing through A, B and C is unique. In fact, the plane passing through A, B and C contains AB and AC, and is unique by Axiom C_3.

Chapter XV

INVESTIGATION OF EUCLIDEAN GEOMETRY AXIOMS

1. Preliminaries

In connection with the axiomatic construction of Euclidean geometry, three questions naturally arise, viz.,

1. Is the axiom system adopted consistent, i.e., can two mutually exclusive corollaries not be derived by logical argument?
2. Is the axiom system complete, i.e., can it not be completed with new axioms consistent with, and not following from the already adopted?
3. Are the adopted axioms independent, i.e., do certain axioms not follow from the others?

The solution to these problems, given in the present chapter, is closely related to the construction of concrete models of an axiom system. A model consists in the indication of quantities of three kinds of arbitrary nature, symbolically named "points", "straight lines" and "planes", and relations among them symbolically expressed by the terms "incident", "between" and "measure", for which the axioms are fulfilled due to their concrete character.

As a matter of fact, the basic notions of geometry are not defined, and everything we know of them is expressed in axioms. Therefore, all our conclusions regard quantities of arbitrary nature, provided the axioms are fulfilled for them and for the relations among them (which can also be very much different from the visual imagery).

To prove the axiom system consistency is to show that at least one of its models exists. To prove that a given axiom is independent means to indicate a model in which all the other axioms except the given one hold. Finally, the proof that some or other axiom system is complete can be performed by showing the isomorphism of all models, i.e., establishing such a one-to-one correspondence between their points, straight lines and planes that the corresponding elements are in similar relations.

2. Cartesian Model of Euclidean Geometry

We now indicate one of the Euclidean geometry models called *Cartesian.* For simplicity, we will construct this model on the plane. It can be easily seen, however, that a similar construction is also valid for a system in space.

We call any pair of real numbers x, y taken in order (x, y) a *point*, and the numbers themselves its *coordinates.* The set of all points whose coordinates satisfy a linear equation

$$ax + by + c = 0, \quad a^2 + b^2 \neq 0$$

is called a *straight line.* The equation is called the *equation of the straight line.* The straight lines $x = 0$ and $y = 0$ are called the *coordinate axes*, whereas the point (0, 0) the *origin.*

We will say that a point *belongs* to a straight line if it is one of its points, i.e., its coordinates satisfy the equation of the straight line.

We show that, with such a concrete understanding of basic concepts, the axioms of incidence hold for Euclidean geometry.

Axiom I_1 which is valid here states that one, and only one, straight line can be drawn through two points. In fact, let (x_1, y_1) and (x_2, y_2) be the two given points.

The straight line determined by the equation

$$(x - x_1)(y_2 - y_1) - (y - y_1)(x_2 - x_1) = 0$$

passes through them, since their coordinates satisfy it. To prove its uniqueness, we assume that two straight lines

$$ax + by + c = 0, \quad a_1x + b_1y + c_1 = 0$$

pass through (x_1, y_1) and (x_2, y_2). Since these two simultaneous equations have two solutions x_1, y_1 and x_2, y_2, they are dependent, i.e., different only by a multiplier, and the straight lines coincide.

Axiom I_2 which also holds here states that at least two points are in each straight line, and that there exist three points not in one straight line.

In fact, let

$$ax + by + c = 0$$

be the equation of a straight line. Then at least one of the coefficients a, b, say, b, is other than zero. We take two arbitrary numbers x_1 and x_2 $(x_1 \neq x_2)$, and find y_1 and y_2 by the formulas

$$y_1 = -\frac{ax_1 + c}{b}, \quad y_2 = -\frac{ax_2 + c}{b}.$$

The points (x_1, y_1) and (x_2, y_2) lie in our straight line.

To prove the existence of three points not lying in the same straight line, we take (0, 0), (0, 1) and (1, 0). In fact, assume that they are in a certain line $ax + by + c = 0$. Substituting their coordinates in the equation, we obtain consecutively that $c = 0$, $b = 0$ and $a = 0$. However, $a^2 + b^2$ must be other than zero, and the contradiction proves the thorem.

3. "Betweenness" Relation for Points in a Straight Line. Verification of the Axioms of Order

We now define the term "between" for the points in a straight line. Let $ax + by + c = 0$ be the equation of a straight line, and (x_1, y_1), (x_2, y_2) and (x_3, y_3) three points in it. In the case where $b \neq 0$, we will say that (x_3, y_3) is between (x_1, y_1) and (x_2, y_2) if the differences $x_1 - x_3$ and $x_3 - x_2$ have the same signs, i.e., the number x_3 is between x_1 and x_2. For $a \neq 0$, we will say that (x_3, y_3) is between (x_1, y_1) and (x_2, y_2) when the differences $y_1 - y_3$ and $y_3 - y_2$ have the same signs. To make the given definition correct, it is required that both defining methods should be equivalent if $a \neq 0$ and $b \neq 0$.

We now prove this equivalence.

If $b \neq 0$, then

$$y_1 = -\frac{ax_1 + c}{b}, \quad y_2 = -\frac{ax_2 + c}{b}, \quad y_3 = -\frac{ax_3 + c}{b}$$

$$y_1 - y_3 = -\frac{a}{b}(x_1 - x_3), \; y_3 - y_2 = -\frac{a}{b}(x_3 - x_2).$$

We see that if $x_1 - x_3$ and $x_3 - x_2$ have the same signs, then $y_1 - y_3$ and $y_3 - y_2$ also have the same signs, and definition equivalence is thus proved.

We now verify that the axioms of order hold. Axiom II_1 states that one, and only one, point of three in a straight line is between the other two. Let $ax + by + c = 0$ be the equation of the straight line, and (x_1, y_1), (x_2, y_2), (x_3, y_3) three points in it. Assume that $b \neq 0$ in the equation. It follows that x_1, x_2, x_3 are all different. Indeed, if $x_1 = x_2$, then

$$y_1 = -\frac{ax_1+c}{b} = -\frac{ax_2+c}{b} = y_2,$$

i.e., (x_1, y_1) and (x_2, y_2) coincide, whereas we mean three different points. Thus, all x_1, x_2, x_3 are different. Place them in ascending order. For definiteness, let $x_1 < x_2 < x_3$. Then the differences $x_2 - x_1$ and $x_3 - x_2$ have the same signs; therefore, (x_2, y_2) is between (x_1, y_1) and (x_3, y_3). The differences $x_3 - x_2$ and $x_1 - x_3$ have different signs. Hence, (x_3, y_3) does not lie between (x_1, y_1) and (x_2, y_2). The differences $x_1 - x_3$ and $x_2 - x_1$ also have different signs. Therefore, (x_1, y_1) does not lie between (x_2, y_2) and (x_3, y_3) either. Thus, of three points in a straight line, one, and only one, lies between the other two.

We now verify that plane-separation Axiom II_2 also holds. Let $ax + by + c = 0$ be the equation of the straight line in question. We will say that a point (x, y) in the plane, not incident with $ax + by + c = 0$, is in the first half-plane if $ax + by + c > 0$, and in the second if $ax + by + c < 0$. The axiom states that if two points A_1 (x_1, y_1) and A_2 (x_2, y_2) lie in the same half-plane, then the line segment A_1A_2 does not meet the straight line. If they are in different half-planes, then the line segment does intersect the straight line.

We show that our plane separation into two half-planes possesses this property. In fact, let $\alpha x + \beta y + \gamma = 0$ be the equation of the straight line joining A_1 to A_2. Suppose, for definitiness, that $\beta \neq 0$. Then $x_1 < x < x_2$ or $x_2 < x < x_1$ for all points (x, y) of the line segment A_1A_2. Substitute their coordinates x and $y = -\frac{1}{\beta}(\alpha x + \gamma)$ in $ax + by + c$. We obtain a linear function of x: $f(x) = c_1x + c_2$. If A_1 and A_2 are in the same half-plane, then $f(x_1)$ and $f(x_2)$ have the same signs; therefore, $f(x)$ preserves sign in the whole interval (x_1, x_2), which means that A_1A_2 does not intersect $ax + by + c = 0$. However, if A_1 and A_2 are in different half-planes, then $f(x_1)$ and $f(x_2)$ have different signs; therefore, $f(x)$ vanishes in (x_1, x_2), and A_1A_2 intersects the straight line. The case $\beta = 0$ (then $\alpha \neq 0$) is considered similarly. Thus, the axioms of order do hold in the Cartesian model.

4. Length of a Segment. Verification of the Axiom of Measure for Line Segments

The number

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

is called the *distance* between two points (x_1, y_1) and (x_2, y_2) in the Cartesian model.

The *length* of a line segment is the distance between its ends.

To verify that the axiom of measure for line segments (Axiom III_1) holds in the Cartesian model, we notice, first of all, that each segment has certain length greater than zero. Let A (x_1, y_1) and B (x_2, y_2) be two points in a straight line, and C (x_3, y_3) a point between them and in the same straight line. We prove that the length of the line segment AB equals the sum of those of the segments AC and BC. Let $y = px + q$ be the straight line equation. Since C is between A and B, either $x_1 < x_3 < x_2$ or $x_1 > x_3 > x_2$. E.g., let $x_1 < x_3 < x_2$.

We have

$$y_1 = px_1 + q, \quad y_2 = px_2 + q, \quad y_3 = px_3 + q.$$

The length of ab equals

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_2 - x_1)^2 + (px_2 - px_1)^2}$$
$$= (x_2 - x_1)\sqrt{1 + p^2}.$$

Similarly, that of AC is $(x_3 - x_1)\sqrt{1 + p^2}$, and $(x_2 - x_3)\sqrt{1 + p^2}$ of BC. We see that the length of AB equals the sum of those of AC and BC; thus, Axiom III_1 holds in the Cartesian model.

For the distances between points in the Cartesian model, the *triangle inequality* is valid. Viz., *the distance between two points is not greater than the sum of their distances from a third point, and is necessarily less if these three points are not in the same straight line.*

Proof. Let a, b, c and d be any four non-negative numbers.

We have

$$a^2d^2 + b^2c^2 \geqslant 2abcd,$$

equality occurring only if $ad = bc$. Add $a^2c^2 + b^2d^2$ to both sides, and take the square root.

We obtain

$$\sqrt{(a^2 + b^2)(c^2 + d^2)} \geqslant ac + bd.$$

Doubling, we then add $a^2 + b^2 + c^2 + d^2$ to both sides, and again take the square root.

We get

$$\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} \geqslant \sqrt{(a + c)^2 + (b + d)^2}.$$

Now, we put $a = x_3 - x_1$, $b = y_3 - y_1$, $c = x_2 - x_3$, $d = y_2 - y_3$. Then the distance between (x_1, y_1) and (x_2, y_2) is on the right, whereas the sum of their distances from (x_3, y_3) on the left. Inequality turns into equality only if $ad = bc$, or if

$$(x_3 - x_1)(y_2 - y_3) = (x_2 - x_3)(y_3 - y_1),$$

i.e., our points are in the same straight line

$$(x_3 - x)(y_2 - y_3) = (x_2 - x_3)(y_3 - y),$$

the coordinates of any of them satisfying the equation.

A *motion* in the Cartesian model is a transformation given by formulas of the form

$$x' = ax + by + c,$$
$$y' = -bx + ay + d,$$

where the constants a and b are such that $a^2 + b^2 = 1$. We see by straightforward verification that motions form a group, which means that a transformation inverse to a motion is a motion, two motions performed one after the other also yield a motion, and the identity transformation ($x' = x$, $y' = y$) is again a motion.

It is verified immediately that a *motion preserves distances between points.*

It follows from the triangle inequality that a motion transforms straight lines into straight lines, half-lines into half-lines, and line segments into line segments.

5. Measure of Angles in Degrees. Verification of Axiom III_2

We define the *measure of angles in degrees* in the Cartesian model. First of all, we assume that the measure of a straight angle is 180°. Consider an angle at the origin with sides in the half-plane $x > 0$ and the equations

$$y = k_1 x, \quad y = k_2 x \quad (x > 0).$$

Then

$$\theta = \frac{180}{\pi} \left| \int_{k_1}^{k_2} \frac{dt}{1+t^2} \right|^*$$

is the measure of our angle in degrees.

We now define the measure of an angle in degrees if it is in general position. Let ABC be an angle other than straight. Let $ax + by + c = 0$ be the equation of the straight line AC. Without loss of

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*Here, we understand $\int_{-\infty}^{\infty} \frac{dt}{1+t^2}$ by π.

generality, we can assume that $a^2 + b^2 = 1$. The motion given by the formulas

$$x' = ax + by + c$$
$$y' = -bx + ay$$

transforms AC into the y-axis. Then, applying the motion given by $\pm x' = x + \alpha$, $y' = y + \beta$, we can send the vertex of the angle B into the origin by carrying out these two motions one after the other with a convenient choice of α and β. Meanwhile, AC will be transformed into a straight line $x = \text{const}$. We can shift AC into the half-plane $x > 0$ by the choice of sign in the second motion formula. Thus, the angle ABC is transformed by a motion into an angle with vertex at the origin and sides in the half-plane $x > 0$. It is the measure of this angle in degrees that we take for the angle ABC.

To see that the above definition of the measure of an angle ABC in degrees is correct, we have to prove its independence of a motion transforming ABC into an angle in the indicated position. To show that this is really so, we suppose the angle ABC is transformed into an angle A_1OC_1, and into an angle A_2OC_2 by another motion. Since all motions form a group, the angle A_1OC_1 is transformed by a motion into the angle A_2OC_2.

Let $y = k_1x$, $y = k_2x$ $(x > 0)$ be the equations of the sides of the angle A_1OC_1, and $x' = \alpha x + \beta y$, $y' = -\beta x + \alpha y$ the motion transforming it into the angle A_2OC_2. To find the equations of the sides of the latter angle, we solve the formulas specifying the motion for x and y, and obtain $x = \alpha x' - \beta y'$, $y = \beta x' + \alpha y'$, substituting which in the side equations for the angle A_1OC_1, we obtain those for the angle A_2OC_2, viz.,

$$y' = k_1'x', \quad k_1' = \frac{\alpha k_1 - \beta}{\beta k_1 + \alpha}.$$
$$y' = k_2'x', \quad k_2' = \frac{\alpha k_2 - \beta}{\beta k_2 + \alpha}$$

The measure of the angle A_1OC_1 in degrees is

$$\frac{180}{\pi}\left|\int_{k_1}^{k_2} \frac{dt}{1+t^2}\right|,$$

and that of the angle A_2OC_2

$$\frac{180}{\pi}\left|\int_{k_1'}^{k_2'} \frac{d\tau}{1+\tau^2}\right|.$$

It is easy to see that they are equal. In fact, it suffices to notice that a change of the variable by the formula $\tau = \dfrac{\alpha t - \beta}{\beta t + \alpha}$ transforms one expression into the other, and the definition correctness for the measure of an angle in degrees is thus proved.

Now, suppose we have an angle (ab) and a ray c between its sides, intersecting some line segment with ends on them. Perform a motion under which the angle is transformed into an angle (a_1b_1) with vertex at the origin and sides in the half-plane $x > 0$. The ray c is then transformed into a ray c_1 between the sides of (a_1b_1). Therefore, the verification of Axiom III_2 is reduced to the case where the vertex of the angle is at the origin, and the sides are in the half-plane $x > 0$.

For check, we suppose that $y = kx$ $(x > 0)$ is a half-line between the sides of the angle $y = k_1x$, $y = k_2x$ $(x > 0)$, which intersect the straight line $x = 1$ at the points $(1, k_1)$ and $(1, k_2)$. The half-line meets the line segment with the ends at these points; therefore, k is between k_1 and k_2.

We have

$$\frac{180}{\pi}\int_{k_1}^{k}\frac{dt}{1+t^2}+\frac{180}{\pi}\int_{k}^{k_2}\frac{dt}{1+t^2}=\frac{180}{\pi}\int_{k_1}^{k_2}\frac{dt}{1+t^2}.$$

Since both addends on the left-hand side have the same signs,

$$\frac{180}{\pi}\left|\int_{k_1}^{k}\frac{dt}{1+t^2}\right|+\frac{180}{\pi}\left|\int_{k}^{k_2}\frac{dt}{1+t^2}\right|=\frac{180}{\pi}\left|\int_{k_1}^{k_2}\frac{dt}{1+t^2}\right|,$$

which means that the measure of the angle in degrees is equal to the sum of those of the angles formed by $y = kx$ $(x > 0)$ and its sides. Thus, Axiom III_2 holds in the Cartesian model.

6. Validity of the Other Axioms in the Cartesian Model

We now verify that Axiom IV of existence of a triangle congruent to a given one holds for a given disposition with respect to a half-line. Let ABC be the given triangle, and PQ the half-line. It is required to prove the existence of a triangle $A_1B_1C_1$ congruent to the triangle ABC, so that the vertex A_1 coincides with the origin of PQ, the vertex B_1 is in this half-line, and the vertex C_1 in the given half-plane with respect to PQ.

Let $ax + by + c = 0$ be the equation of the straight line AB. Without loss of generality, we can assume that $a^2 + b^2 = 1$. The motion specified by the formulas

$$\pm x' = ax + by + c$$
$$\pm y' = -bx + ay + \lambda$$

transforms AB into the y-axis. We can send the point A into the origin, the point B onto the half-axis $y > 0$, and transform the half-plane containing the point C into the half-plane $x > 0$ with respect to AB by a convenient choice of λ and the signs of x' and y'. We denote the obtained motion by S.

Now, let $a_1x + b_1y + c_1 = 0$ be the equation of the straight line PQ. The motion given by the formulas

$$\pm x' = a_1x + b_1y + c_1$$
$$\pm y' = -b_1x + a_1y + \mu$$

transforms PQ into the y-axis. By a convenient choice of μ and the signs of x' and y', we can send the point P into the origin, the point Q onto the half-axis $y > 0$, and transform the given half-plane with respect to PQ into the half-plane $x > 0$. We denote the obtained motion by H, and the inverse by H^{-1}.

Perform the motions S and H^{-1} one after the other. The triangle ABC is then transformed into a triangle $A_1B_1C_1$ with the given disposition relative to the half-line PQ. It remains to prove that they are congruent. Since the motion preserves distances, their corresponding sides are congruent. We now show that the corresponding angles are congruent. To find the measure of the angle $A_1B_1C_1$ in degrees, we transform it by a motion into an angle A_2OC_2 with vertex at the origin and sides in the half-plane $x > 0$.

We take as the measure of the angle $A_1B_1C_1$ in degrees that of the angle A_2OC_2, for which we have a formula (see Sec. 5). Since the angle ABC is transformed by a motion into the angle $A_1B_1C_1$, and the latter into the angle A_2OC_2, the angle ABC is transformed by a motion into the angle A_2OC_2, and, therefore, has the same measure as the angle $A_1B_1C_1$. Thus, they are congruent. The congruence of the other corresponding angles of the triangles ABC and $A_1B_1C_1$ is proved similarly. The validity of Axiom IV in the Cartesian model is proved.

That Axiom V of the existence of a line segment of any given length d holds in the Cartesian model is sufficiently obvious. In fact, the line segment with ends at the points $(0, 0)$ and $(d, 0)$ has length $\sqrt{(d-0)^2 + (0-0)^2} = d$.

To verify that the parallel axiom holds, we prove that, in the Cartesian model, not more than one straight line can be drawn parallel to a straight line $ax + by + c = 0$ through an outside point (x_0, y_0). Assume that there are two such lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ passing through (x_0, y_0), and parallel to the given line, i.e., never meeting it. Then both pairs of simultaneous equations

$$a_1x + b_1y + c_1 = 0 \qquad a_2x + b_2y + c_2 = 0$$
$$ax + by + c = 0 \qquad ax + by + c = 0$$

are inconsistent, or have no solutions. Therefore,

$$\begin{vmatrix} a_1 & b_1 \\ a & b \end{vmatrix} = 0, \quad \begin{vmatrix} a_2 & b_2 \\ a & b \end{vmatrix} = 0.$$

Hence,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

Since the simultaneous equations

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

have a solution $x = x_0$, $y = y_0$, they are dependent, and differ only by a multiplier, which means that the straight lines are coincident contrary to the assumption. Thus, the validity of this axiom in the Cartesian model is proved.

7. Consistency and Completeness of the Euclidean Geometry Axiom System

The system of axioms for any theory T, and, in particular, for Euclidean geometry, is consistent if it admits at least one model R.

In fact, if two mutually exclusive corollaries could be derived from the system of axioms for T, then this would be also possible for R. Since the validity of each statement in R, corresponding to an axiom in T, seems doubtless due to the nature of the objects in R and the relations among them, to obtain two such corollaries in R is impossible. Hence the impossibility to come to a contradiction in T.

We have already constructed one model of Euclidean geometry, viz., Cartesian. The method was to indicate a system of objects called points and straight lines, and a system of relations among them, so that all the statements contained in the Euclidean geometry axiom system were valid. The conclusion that they are, in fact, true was made on the basis of the corresponding theorems related to the theory of real numbers. Since they are, eventually, deducible from the axioms for arithmetic, we can warrant the Cartesian model construction, provided that the arithmetic axiom system is consistent. Thus, we obtain a solution to the problem of the Euclidean geometry axiom system consistency in the following form.

The Euclidean geometry axiom system is consistent if the arithmetic axiom system is.

We now turn to the problem of an axiom system completeness. Consider two models R' and R'' of a certain theory T. We call them *isomorphic* if their elements can be put into a one-to-one correspondence preserving the axiomatically determined relations.

An axiom system T is said to be *complete* if no new axioms can be added, which do not follow from those of T, and are consistent with them. We certainly assume that the new axioms do not introduce any new relations, and that the new system so formed admits a model. The problem of an axiom system completeness is intimately related to that of an isomorphism of all its models. Viz., *if all models of an axiom system T are isomorphic, then it is complete.*

Indeed, let an axiom system T be incomplete, which means that there is a certain statement a not deducible from the axioms of T, and consistent with them. Meanwhile, two consistent axiom systems T' and T'' can be formed by adding to T the axiom a or its negation $\bar{a}$.

Let R' and R'' be two models of T' and T'', each of which is, at the same time, that of T. Since a is valid in T', and $\bar{a}$ in T'', these models of T are not isomorphic, and the statement is thus proved.

The Euclidean geometry axiom system is complete, i.e., no new axioms regarding points, straight lines or planes and the relations among them, determined by the axioms, can be added, so that they do not follow from the already adopted axioms, and are consistent with them.

For proof, it suffices to establish that all models of Euclidean geometry are isomorphic. Since it is obvious that two models isomorphic to a third are isomorphic to each other, it suffices to prove the isomorphism of all models of the Cartesian one.

We now establish such an isomorphism. Let R be any model of Euclidean plane geometry. Introduce rectangular Cartesian coordinates on the plane as is done in analytic geometry (Part One). Each straight line on the plane is known to be given by a linear equation $ax + by + c = 0$, and each such equation to be that of a certain straight line.

It is also known that the mutual disposition of three points in a straight line, expressed by the term "between", leads to a certain relation among the point coordinates. Viz., if a point (x, y) is between (x_1, y_1) and (x_2, y_2), then either x is between x_1 and x_2, or y is between y_1 and y_2 or both.

For the distance between (x_1, y_1), (x_2, y_2) in rectangular Cartesian coordinates, the formula $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ is deduced, and the concept of motion introduced as of a distance-preserving transformation, for which the formulas

$$\begin{aligned} x' &= ax + by + c \\ \pm y' &= -bx + ay + c_1 \end{aligned} \qquad (a^2 + b^2 = 1)$$

are obtained. It is proved, meanwhile, that measures of angles in degrees are preserved under motions.

All the above is well-known from analytic geometry (see Part One).

Associate a point (x, y) in the Cartesian model with a point in the model R with coordinates x and y, while a straight line $ax + by + c = 0$ in the Cartesian model with that in R, given by the same equation. We assert that this one-to-one correspondence between the Cartesian model points and straight lines and those in R is an isomorphism. In fact, if a point A is in a straight line a in the Cartesian model, and A', a' are the corresponding point and straight line in R, then A' is in a'. If three points A, B, C in the Cartesian model are on a straight line, and B is between A and C, then the corresponding points A', B', C' are positioned similarly in R, i.e., B' is between A' and C'. The corresponding line segments of the Cartesian model and R are equally long, as expressed by the same formula in terms of the end-point coordinates.

We show that the corresponding angles in the Cartesian model and R are of the same measure in degrees. First of all, we notice that motions are given by the same formulas, and preserve measures in degrees. Transform by a motion the corresponding angles of our models into those with sides

$$y=x\tan\theta_1,\quad y=x\tan\theta_2,\quad x>0,\quad -\frac{\pi}{2}<\theta_1,\quad \theta_2<\frac{\pi}{2}.$$

Then the measure of the angle in degrees is $|\theta_2-\theta_1|\dfrac{180^\circ}{\pi}$ in R, whereas that of the corresponding angle

$$\frac{180^\circ}{\pi}\left|\int\limits_{\tan\theta_1}^{\tan\theta_2}\frac{dt}{1+t^2}\right|=\frac{180^\circ}{\pi}\tan^{-1}(\tan\theta_2)$$

$$-\tan^{-1}(\tan\theta_1)|=|\theta_2-\theta_1|\frac{180^\circ}{\pi}.$$

We see that the measures of the corresponding angles in degrees are the same in both models.

Thus, the established correspondence between points and straight lines of the Cartesian model and R is an isomorphism, whence all the Euclidean geometry axiom system models are isomorphic; therefore, the axiom system is complete.

8. Independence of the Axiom of Existence of a Line Segment of Given Length

An axiom a of a theory T with axiomatic construction is said to be *independent* if it cannot be derived as a corollary to the other axioms. The usual method for the proof of independence of some or other axiom a is in the construction of a model R of the system of axioms for T without a, in which a would not be valid. If such a model is constructed, then a is independent.

Indeed, if a were obtained as a corollary to the remaining axioms, then the statement a would also hold in R, which is contrary to its construction.

It is just in this way that we prove the independence of the axiom of existence of a line segment of given length from the remaining Euclidean geometry axioms. We prove the following theorem.

The axiom of existence of a line segment of given length is independent, i.e., it cannot be obtained as a corollary to the other Euclidean geometry axioms.

Proof. Let G be a set of real numbers, containing all rational, and also all those obtained from the rational by a finite number of additions, subtractions, multiplications, divisions and square-root operations. It is evident that the sum, difference, product, quotient of two numbers from G, and also the square root of any non-negative number, are again in G. It is known also that the numbers from G do not exhaust all real numbers. Moreover, G is at most countable, whereas the set of all real numbers is uncountable.

We now construct the Cartesian model of Euclidean geometry as before, but only with the elements from G.

Thus, we call a pair of numbers (x, y) from G a *point*, and the set of points satisfying any linear equation $ax + by + c = 0$ with coefficients in G a *straight line*. The *relation of order* for points in a straight line is, as before, defined in terms of the point coordinates. A *motion* is a transformation given by formulas of the form

$$x' = ax + by + c, \quad \pm y' = -bx + ay + d \quad (a^2 + b^2 = 1)$$

with coefficients in G. The *length* of a line segment and the *measure* of an angle in degrees are defined as before verbatim.

We can now start verifying the axioms. All the proofs given in connection with the Cartesian model of Euclidean geometry (see Secs. 2-8) are repeated verbatim except that for the axiom of existence of a line segment of given length, since it does not hold at all.

Indeed, the length of any segment is the distance between its endpoints, and is determined by the formula

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Since the values x_1, y_1, x_2, y_2 are in G, the length of the segment is also in G. The axiom of existence of a line segment of given length states that, for any real number d, there is a line segment of length d. Since the numbers from G do not exhaust all real numbers, there is a value d that cannot be the length of any segment. Therefore, the axiom of existence of a line segment of given length does not hold in the model constructed, and, hence, this axiom does not depend on the others in Euclidean geometry.

9. Independence of the Parallel Axiom

The parallel axiom for Euclidean geometry is independent, i.e., cannot be deduced from the other axioms.

Proof. According to the general method for the axiom independence proof, it suffices to construct such a model of the Euclidean geometry axiom system without the parallel axiom that it does not hold. We shall now construct such a model, for simplicity confining ourselves to the system of axioms for the plane.

By a *point*, we understand any point in the Euclidean plane inside the unit circle

$$x^2 + y^2 < 1,$$

and by a *straight line* any chord of the circle. The *incidence* and *order relations* are understood to be the same as in Euclidean geometry.

By a motion, we mean a transformation of the forms

$$\begin{aligned} x' &= ax + by \\ \pm y' &= -bx + ay \end{aligned} \qquad (a^2 + b^2 = 1) \tag{*}$$

or

$$x' = \frac{x\sqrt{1-\beta^2}}{1+\beta y}, \quad y' = \frac{y+\beta}{1+\beta y}, \quad |\beta| < 1 \tag{**}$$

and also any transformation obtained by performing two in (*) and (**) one after the other.

It is obvious that the motions form a group. The transformations (*) and (**) send the circle $x^2 + y^2 < 1$ into itself, it being obvious for (*), and easily verifiable for (**), since $x'^2 + y'^2 < 1$. Hence, *any motion transforms the circle $x^2 + y^2 < 1$ into itself.*

It is seen by straightforward check that any motion can be given by formulas of the form

$$x' = \frac{a_1x + b_1y + c_1}{ax + by + c}, \quad y' = \frac{a_2x + b_2y + c_2}{ax + by + c} \tag{***}$$

(denominators being the same).

Hence, *by a motion, straight lines are transformed into straight lines.* In fact, let a straight line h be given by an equation $Ax + By + C = 0$. Solving (***) for x, y, and substituting them in $Ax + By + C = 0$, we obtain a linear equation $A'x' + B'y' + C' = 0$ in x' and y', which means that h is transformed into a straight line h' with this equation.

A motion preserves the order of points in a straight line. In fact, for definiteness, let $B \neq 0$ and $B' \neq 0$ in the equations of two straight lines h and h'. Substituting

$$y = -\frac{Ax + C}{B}$$

in the first formula of (***), we obtain

$$x' = \frac{\alpha x + \beta}{\gamma x + \delta},$$

which establishes the relation between the coordinate x of a point in h and the coordinate x' of the corresponding point in h', viz.,

$$\frac{dx'}{dx} = \frac{\alpha\delta - \beta\gamma}{(\gamma x + \delta)^2}.$$

We see that dx'/dx preserves sign. Therefore, x' is a monotonic function of x, which means that if, for three points $x_1 < x_2 < x_3$ in h, then either $x'_1 < x'_2 < x'_3$ or $x'_1 > x'_2 > x'_3$ in h' for the corresponding points, and motions preserve the order of points in straight lines.

Since, under a motion, straight lines are transformed into straight lines, and the order of points is preserved, *line segments are transformed into line segments, and rays into rays.*

We define the distance between two points A (x_1, y_1) and B (x_2, y_2) as follows. The straight line AB meets the circumference $x^2 + y^2 = 1$ at two points C (x_3, y_3) and D (x_4, y_4). We call the value

$$\left| \ln \left(\frac{x_3 - x_1}{x_3 - x_2} \div \frac{x_4 - x_1}{x_4 - x_2} \right) \right|$$

the *distance* between A and B if $x_1 \neq x_2$, or a similar expression, replacing x by y, if $y_1 \neq y_2$. In the case where $x_1 \neq x_2$, and $y_1 \neq y_2$, we can use any formula with the same result. As a matter of fact, for $x_1 \neq x_2$, $y_1 \neq y_2$, $a \neq 0$ and $b \neq 0$ in the equation $ax + by + c = 0$ of AB. Hence, $x = -\frac{by+c}{a}$. If we substitute y for x by means of the latter expression, then we obtain

$$\left| \ln \left(\frac{y_3 - y_1}{y_3 - y_2} \div \frac{y_4 - y_1}{y_4 - y_2} \right) \right|.$$

Motions preserve distances. Indeed, let a motion send two points A (x_1, y_1) and B (x_2, y_2) into A' (x'_1, y'_1) and B' (x'_2, y'_2). The distance between A and B is

$$d = \left| \ln \left(\frac{x_3 - x_1}{x_3 - x_2} \div \frac{x_4 - x_1}{x_4 - x_2} \right) \right|,$$

whereas that between A' and B'

$$d' = \left| \ln \left(\frac{x'_3 - x'_1}{x'_3 - x'_2} \div \frac{x_4 - x'_1}{x'_4 - x'_2} \right) \right|.$$

The relation between the coordinate x of a point in AB and the coordinate x' of the corresponding point in $A'B'$ is known to be established by the formula

$$x' = \frac{\alpha x + \beta}{\gamma x + \delta}.$$

Substituting it in the formula for d', we obtain after a simple calculation that $d' = d$, i.e., the distance between two points is preserved under a motion.

The *measure* of an angle in degrees is defined as for the Cartesian model (see Sec. 5), with the only difference that a motion is understood here in the sense of the above definition.

The measure of an angle in degrees is preserved under a motion.

We should now verify that the axioms hold in the model constructed. That the axioms of incidence and order are valid is quite obvious. That the axiom of measure for line segments holds follows from the law of logarithms

$$\ln ab = \ln a + \ln b.$$

That the axiom of measure is true for angles is verified verbatim as in the Cartesian model, the motion understood in the sense of the above definition.

The verification of the axiom of existence of a triangle congruent to a given one is done as for the Cartesian model.

To verify the axiom of existence of a line segment of given length, consider a segment with ends at points $(0, 0)$ and $(x, 0)$, $\left|\ln \frac{1}{1-x}\right|$ in length. It is evident that any number d can be thus obtained by a convenient choice of x.

In a word, all the axioms for Euclidean geometry hold in the constructed model except the parallel axiom. In fact, through a given point in a circle, we can draw an infinite number of chords not intersecting a given one. It is the construction of this model that proves the independence of the parallel axiom from the other Euclidean geometry axioms.

10. Lobachevskian Geometry

We have proved that the parallel axiom does not depend on the other Euclidean geometry axioms. It follows that the former can be replaced by its negation in the Euclidean geometry axiom system. The axiom system so formed is also consistent, since it does admit a model (see Sec. 9). The corresponding geometry is said to be Lobachevskian. Thus, Lobachevskian geometry axiom system consists of that for Euclidean geometry, the parallel axiom replaced by the Lobachevskian. Viz., *through every point not in a straight line, there are at least two straight lines not intersecting it.*

It turns out that *the system of axioms for Lobachevskian geometry is complete*; therefore, the latter can be studied in any of its models. The one obtained in the previous section is due to F. Klein, and often called the *Klein model* of Lobachevskian geometry.

In Lobachevskian geometry, a whole pencil of straight lines not intersecting a given straight line passes through a given outside point. Its extreme lines are said to be *parallel* to the given in the sense of Lobachevsky. Straight lines parallel in the sense of Lobachevsky are represented in the Klein model as chords with a common end.

We now clarify how perpendicular straight lines are represented in the Klein model. If they intersect at the centre of the circle, then their perpendicularity in the sense of Lobachevsky implies the usual perpendicularity in the sense of Euclidean geometry (Fig. 115*a*). In the case where straight lines do not meet at the centre, their perpendicularity in the sense of Lobachevsky means that the tangents at the ends of one chord intersect at the extension of the other (Fig. 115*b*). We give the proof later.

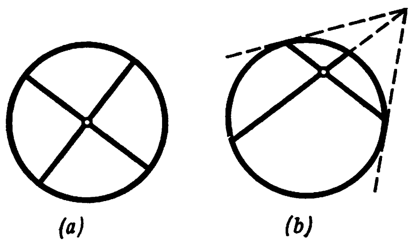

Fig. 115

Knowing how to find the distance between points in the Klein model, we can find the distance ds between two arbitrarily near points (x, y) and $(x + dx, y + dy)$, or the *linear element of the Lobachevsky plane*. Omitting the necessary argumentation, we only give the final result, viz.,

$$ds^2 = c\,\frac{dx^2 + dy^2 - (x\,dy - y\,dx)^2}{1 - x^2 - y^2},$$

where c is a positive constant.

Considering the linear element ds^2 as that of a surface in Euclidean space, we clarify what is characteristic of the surface. Accordingly, we find its Gaussian curvature. By the Gauss formula (Ch. XI, Sec. 7) for Gaussian curvature in terms of the linear element coefficients, we get $K = -c$. Thus, *the Lobachevsky plane is locally isometric to a surface of constant negative curvature*, and we obtain another model of Lobachevskian geometry. This was constructed by E. Beltrami.

We now establish what straight lines are in the Beltrami model. Their characteristic property is that they are the shortest. Since a mapping of the Lobachevsky plane onto a surface of constant negative curvature is isometric, *the Lobachevsky straight lines on a surface of constant negative curvature in the Beltrami model are geodesic lines.* The distance between points in the Beltrami model is the length of the geodesic segment joining them.

What is a motion in the Beltrami model? It is a distance-preserving, or isometric, transformation of the surface.

A good number of the Euclidean geometry theorems also hold in Lobachevskian geometry, e.g., on the sum of supplementary angles and congruence of vertically opposite angles, tests of congruence of triangles, etc. Still, there are certain theorems in Lobachevskian geometry, not being true for Euclidean geometry. We illustrate by several examples.

In Lobachevskian geometry, the sum of the angles of a triangle is less than 180°.

In Lobachevskian geometry, there exist no triangles of arbitrarily large area.

In Lobachevskian geometry, there are no similar or congruent triangles.

We now prove them by means of the Beltrami model.

By the Gauss-Bonnet theorem,

$$\alpha + \beta + \gamma - \pi = K\sigma, \tag{*}$$

where α, β, γ are the angles of a triangle (in radians), σ is its area, and K a negative constant. Since $K < 0$, we have $\alpha + \beta + \gamma < \pi$, and the first theorem is thus proved.

To prove the second statement, we remember that

$$\sigma = \frac{\alpha+\beta+\gamma-\pi}{K}.$$

Since α, β, $\gamma > 0$, we have $\sigma < \pi/|K|$, i.e., the area of any triangle is bounded by $\pi/|K|$, and the second theorem is also proved.

For the proof of the third theorem, we assume that

$$\angle A = \angle A_1, \quad \angle B = \angle B_1, \quad \angle C = \angle C_1$$

$$A_1B_1 = kAB, \quad A_1C_1 = kAC, \quad B_1C_1 = kBC, \quad k < 1$$

for two triangles ABC and $A_1B_1C_1$. Move the triangle $A_1B_1C_1$ so that its vertex A_1 coincides with A, the vertex B_1 is on the side AB, and the vertex C_1 is on the side AC. The triangle $A_1B_1C_1$ is then inside the triangle ABC, and, therefore, is of less area. But the area of a triangle is expressed in terms of its angle-sum by the formula (*), whereas the corresponding angles of our triangles are congruent; a contradiction, and the third theorem is proved as well.

We now give another model of Lobachevskian geometry, the Poincaré model. Project the Klein circle $x^2 + y^2 < 1$ onto the hemisphere $x^2 + y^2 + z^2 = 1$, $z > 0$ by straight lines parallel to the z-axis, and, in turn, the hemisphere onto the yz-plane from the point (1, 0, 0). We then obtain a mapping of the circle $x^2 + y^2 < 1$ onto the half-plane of the yz-plane ($z > 0$).

We now clarify into what the circle chords, i.e., Lobachevsky straight lines, will be transformed. A point (x, y) in the circle is under the first projection sent into the point $(x, y, \sqrt{1 - x^2 - y^2})$

on the hemisphere. To find the trace of (x, y) on the yz-plane under the second projection, consider the projecting line given by the equations

$$\frac{\bar{x}-1}{x-1}=\frac{\bar{y}}{y}=\frac{\bar{z}}{\sqrt{1-x^2-y^2}},$$

and intersecting the yz-plane at the point

$$\bar{x}=0,\ \bar{y}=-\frac{y}{x-1},\ \bar{z}=-\frac{1}{x-1}\sqrt{1-x^2-y^2}.$$

We have

$$\bar{y}^2+\bar{z}^2=\frac{1-x^2}{(x-1)^2}=\frac{1+x}{1-x}.$$

Hence,

$$x=\frac{-1+\bar{y}^2+\bar{z}^2}{1+\bar{y}^2+\bar{z}^2},\quad y=\frac{2\bar{y}}{1+\bar{y}^2+\bar{z}^2}. \qquad (**)$$

Let a chord be given by the equation

$$ax+by+c=0.$$

Substituting the expression for x and y, we obtain the equation of the curve into which the chord is sent under the mapping in question, viz.,

$$a(-1+\bar{y}^2+z^2)+2by+c(1+y^2+z^2)=0,$$

or

$$(c+a)(\bar{y}^2+\bar{z}^2)+2b\bar{y}+(-a+c)=0,\ \bar{z}>0.$$

If $c+a\neq 0$, then it is the equation of a semi-circle with centre on the y-axis. If $c+a=0$, then it is that of a straight line perpendicular to the y-axis (Fig. 116). Thus, the Lobachevsky straight lines are represented in the Poincaré model by semicircles with centres on the half-plane boundary and by straight lines perpendicular to it.

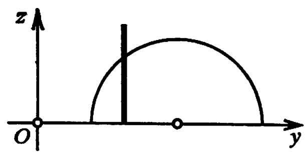

Fig. 116

If, according to (**), the variables $\bar{y}$ and $\bar{z}$ are introduced into the linear element ds^2 of the Lobachevsky plane instead of x and y, then, as computations show, it can be reduced to the form

$$ds^2=\frac{d\bar{y}^2+d\bar{z}^2}{\bar{z}^2}.$$

Since $d\bar{y}^2+d\bar{z}^2$ is the linear element of the yz-plane, a mapping of the Lobachevsky plane onto the Poincaré half-plane is conformal.

In conclusion, we notice that Lobachevsky motions correspond in the Poincaré model to inversions with respect to circles with centres on the boundary of the half-plane, translations parallel to the half-plane boundary, and similarities with respect to centres on the half-plane boundary.

Chapter XVI

PROJECTIVE GEOMETRY

1. Axioms of Incidence for Projective Geometry

Projective geometry originated in the first half of the 19th c., and is related to the name of the French geometer V. Poncelet (1788-1867) who delineated the subject matter of projective geometry, i.e., the properties of figures and of related quantities invariant under any projection.

Projective geometry was also much developed by M. Chasles (1793-1880) and J. Steiner (1769-1863). Thanks to the works of K. Staudt (1798-1867), the science was freed of the concept of metric, foreign to it, and was turned into a discipline only studying the properties of geometric figure disposition.

Projective geometry is constructed on the basis of a system of axioms of incidence, order, and also the axiom of continuity.

Axioms of incidence speak of mutual disposition of points, straight lines and planes, expressed by the term "to be incident". Meanwhile, the agreement remains valid regarding the equivalent expressions indicated in the Euclidean geometry axioms of incidence.

Axiom I_1. For any two points A and B, there is a straight line incident with each of these points.

Axiom I_2. For any two points A and B, there exists not more than one straight line incident with each.

Axiom I_3. There exist at least three points in each straight line. There are at least three points not in one straight line.

Axiom I_4. There is a certain plane α passing through any three non-collinear points A, B and C. There is at least one point in each plane.

Axiom I_5. Not more than one plane passes through any three points not in one straight line.

Axiom I_6. If two points A and B of a straight line a are in a plane α, then each point of the line is in α.

Axiom I_7. If two planes have a point in common, then they have at least one more point in common.

Axiom I_8. There are at least four points not in one plane.

Axiom I_9. Any two straight lines in one plane have a common point.

We see that the axioms of incidence for projective geometry contain those of Euclidean geometry, and differ from the latter only in Axiom I_3 requiring the existence of at least three points in a straight line, and Axiom I_9 stating that any two straight lines in one plane should meet.

Hence, all the corollaries to the axioms of incidence for Euclidean geometry also hold in projective geometry. Axioms I_3 and I_9 permit us to extend the set of the corollaries; in particular, it can be easily proved that

(i) a straight line and a plane always have a point in common,
(ii) two planes have a straight line in common, and
(iii) three planes have a point in common.

2. Desargues Theorem

The most important of the corollaries to the axioms of incidence for projective geometry is the *Desargues theorem* on two sets of three points in perspective.

A *set of three points* is a figure made up of three points not in one straight line, its *vertices*, and three lines joining them pairwise, its *sides*. Two sets of three points ABC and $A'B'C'$ are said to possess a *centre of perspective* S if the vertices A and A', B and B', C and C' are in straight lines passing through S. ABC and $A'B'C'$ are said to possess an *axis of perspective* s if the sides AB and $A'B'$, BC and $B'C'$, AC and $A'C'$ meet at its points.

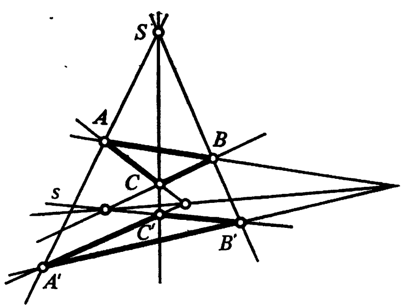

Fig. 117

If two sets of three points ABC and $A'B'C'$ possess an axis of perspective, then they also have a centre of perspective. Conversely, if they have a centre of perspective, then they also have an axis of perspective (see Fig. 117).

Proof. First, we notice that if two corresponding vertices or sides of two sets of three points coincide, then the statement of the theorem is quite obvious. Therefore, we can confine ourselves to the case where the corresponding vertices and sides are different.

To begin with, we assume that the planes σ and σ' of these sets are different. Then the planes intersect in a straight line s, its points exhausting all points common to σ and σ'.

Let the sets have an axis of perspective. Since the sides AB and $A'B'$ meet, but are different, there is one, and only one, plane γ incident with them. The planes α and β incident with the sides BC and $B'C'$, AC and $A'C'$, respectively, are determined similarly. Since σ and σ' are different, α, β and γ are also different, the first two intersecting in CC', β and γ in AA', whereas γ and α in BB'. It follows that the point S common to all the planes is the centre of perspective of the sets.

Let the sets have a centre of perspective. Since AA' and BB' meet, the points A, A', B, B' are in one plane. Therefore, the straight lines AB and $A'B'$ intersect, and, since the planes σ and σ' of these sets are different, the point, where the straight lines meet, is incident with the straight line s in which σ and σ' intersect. It is shown similarly that the sides AC and $A'C'$, BC and $B'C'$ also intersect at s. Therefore, the sets have the axis of perspective s.

Now, let both sets be in one plane σ, and s their axis of perspective. Draw through s a plane σ' other than σ. Such a plane exists, for, by Axiom I_8, there is a point P not in σ, and, by Axiom I_2, there are two points Q and R in s. σ' is incident with P, Q, R, and different from σ by Axiom I_5.

We now take a point O outside σ and σ'. Such a point does exist. In fact, there are four points K, L, M, N not in one plane. At least one of the points is outside σ. Let it be N. Project K, L and M onto σ from N as a centre. The points $\bar{K}$, $\bar{L}$, $\bar{M}$ obtained are not in one straight line. Therefore, in σ, there is a point not in s. The existence of such a point in σ' is proved similarly. By Axiom I_3, the straight line joining these two points possesses at least one more point O lying outside σ and σ'.

Project the set of three points $A'B'C'$ onto σ' from O, obtaining a set of three points $A''B''C''$. The straight line s is the axis of perspective for the sets ABC and $A''B''C''$. Therefore, they have a centre of perspective S (proved). Let $\bar{S}$ be the projection of S onto σ from O as a centre. We assert that $\bar{S}$ is the centre of perspective for ABC and $A'B'C'$.

Indeed, the straight lines AA'', BB'', CC'' meet at S. Consequently, their projections AA', BB', CC' onto σ meet at $\bar{S}$.

Now, let these sets be in one plane σ, and possess a centre of perspective S. Take a point O outside σ. In the straight line OA, there is a point $\bar{A}$ different from A and O. Join it to S with a straight line g, and project A' onto it from O. Denote the projection by $\bar{A}'$. The point S is the centre of perspective for the sets $\bar{A}BC$ and $\bar{A}'B'C'$ possessing an axis of perspective s (proved), whose projection onto σ is the axis of perspective for ABC and $A'B'C'$.

Q.E.D.

3. Completion of Euclidean Space with the Elements at Infinity

The system of axioms for projective geometry is complete. It can be, therefore, studied in any of its models, the simplest and most visual obtained by completing Euclidean space with the elements at infinity, i.e., points, straight lines and planes at infinity, as follows.

First, homogeneous coordinates are introduced. Any four numbers $x_1, x_2, x_3, x_4, x_4 \neq 0$, related to the Cartesian coordinates of a point by

$$x=\frac{x_1}{x_4}, \quad y=\frac{x_2}{x_4}, \quad z=\frac{x_3}{x_4} \qquad (*)$$

are called its *homogeneous coordinates* in Euclidean space.

Thus, the homogeneous coordinates of a point are not determined uniquely. If x_1, x_2, x_3, x_4 are the homogeneous coordinates of a point, then the values $\rho x_1, \rho x_2, \rho x_3, \rho x_4, \rho \neq 0$, are also those of the same point.

With respect to Cartesian coordinates, a plane is given by a linear equation

$$ax + by + cz + d = 0.$$

Substituting x, y and z expressed in terms of homogeneous coordinates, and noticing that $x_4 \neq 0$, we obtain an equation

$$ax_1 + bx_2 + cx_3 + dx_4 = 0,$$

now with respect to homogeneous coordinates.

Thus, *with respect to homogeneous coordinates, a plane is given by a homogeneous linear equation.*

Similarly, we conclude that, *with respect to homogeneous coordinates, a straight line is given by two independent, homogeneous simultaneous linear equations.*

Each set of four numbers $x_1, x_2, x_3, x_4, x_4 \neq 0$, is associated with a certain point in space with Cartesian coordinates x, y, z which can be found by the formulas (*). A set of four numbers with $x_4 = 0$ does not correspond to any point. We will say that they are associated with a *point at infinity*, provided not each is zero. Euclidean space completed with points at infinity is said to be *projective*. A *plane* in a projective space is a set of points whose homogeneous coordinates satisfy a homogeneous linear equation, and a *straight line* a set of points satisfying two independent simultaneous linear equations. With such an agreement, *the passage from Euclidean to a projective space is accompanied by completing each Euclidean straight line with a point at infinity, each plane with a straight line at infinity, and space with a plane at infinity.*

In fact, the set of the points at infinity in space satisfies the equation $x_4 = 0$. It is linear, and, by definition, that of a plane. The points at infinity in a plane satisfy the two simultaneous equations

$$ax_1 + bx_2 + cx_3 + dx_4 = 0, \quad x_4 = 0,$$

by definition, specifying a projective line.

The points at infinity in a straight line are given by the simultaneous equations

$$ax_1 + bx_2 + cx_3 + dx_4 = 0$$
$$a'x_1 + b'x_2 + c'x_3 + d'x_4 = 0, \quad x_4 = 0$$

having a non-trivial solution which is unique up to a constant multiplier. Hence, the passage from a Euclidean straight line to a projective one is accompanied by adding one point at infinity to the former.

If plane problems are considered, then three homogeneous coordinates x_1, x_2 and x_3 are used. Meanwhile, those points for which $x_3 = 0$ are at infinity. In a projective plane, a straight line is given by a homogeneous linear equation

$$ax_1 + bx_2 + cx_3 = 0;$$

in particular, that at infinity by $x_3 = 0$.

4. Topological Structure of a Projective Straight Line and Plane

We now find certain simple forms topologically equivalent to a projective straight line and plane, defining *nearness* in a projective space. We call the set of all points y (y_1, y_2, y_3, y_4) for which $|x_1 - y_1| < \varepsilon$, $|x_2 - y_2| < \varepsilon$, $|x_3 - y_3| < \varepsilon$, $|x_4 - y_4| < \varepsilon$ a *neighbourhood* of a point x (x_1, x_2, x_3, x_4) in a projective space. We assume that a point y is *near* to x if ε is sufficiently small.

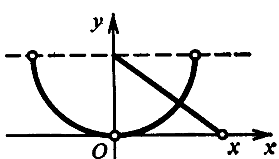

Fig. 118

Take the semi-circle $x^2 + (y - 1)^2 = 1$, $y < 1$, in the xy-plane. The projection of the x-axis as a Euclidean straight line onto the semi-circle from its centre is a topological transformation of a straight line into a semi-circle (Fig. 118). As a projective straight line, the x-axis has the point at infinity (1, 0, 0). The sufficiently distant points of the x-axis, when $|x|$ is large, are near to the point at infinity, since the coordinates 1, 0, $\frac{1}{x}$ are homogeneous, which makes it possible to regard the semi-circle ends as identical, and associate them with the point at infinity in the

x-axis. We then obtain a topological mapping of the projective line onto a closed curve, a semi-circle with coincident ends. Thus, *a projective straight line is topologically equivalent to a closed curve, e.g., circle.*

To find a topologically equivalent form of a projective plane, we take the hemisphere $x^2 + y^2 + (z - 1)^2 = 1$, $z < 1$ (Fig. 119*a*). Repeating the same argument as for a projective straight line, we conclude that the projective xy-plane can be topologically mapped onto the hemisphere if diametrically opposite points of its boundary are regarded as identical. However, in contrast to a projective straight line, it is rather hard to imagine the form obtained, and we

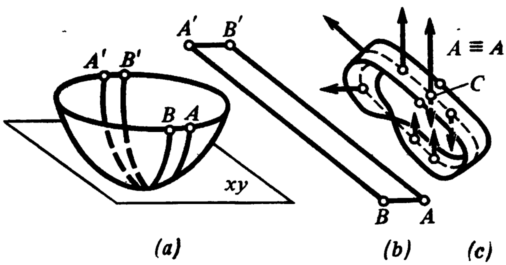

Fig. 119

remove a segment consisting of two half-segments cut off by the planes $x = \varepsilon$ and $x = -\varepsilon$ for small ε (Fig. 119*a*). Since their ends in the hemisphere boundary are assumed to be identical, they all make up a complete segment when taken together.

We now investigate the remaining part of the hemisphere, which is between the planes $x = \pm\varepsilon$. It is not complicated to imagine its topological transformation into a narrow rectangle (Fig. 119*b*) whose sides AB and $A'B'$ are coincident, the point A falling on A', and B on B'. The obtained surface is called a *Möbius strip* (Fig. 119*c*).

Its boundary is made up of the sides AB' and BA' extending each other if the rectangle is glued into a Möbius strip. A Möbius strip is a unilateral surface. If, specifying a direction of the normal to the surface at a point C, we take a non-stop walk along the dotted line, then we come to C again, reversing the normal direction. These properties may be better illustrated by a narrow slip of paper with its narrower sides glued together.

Returning to the problem of a topologically equivalent form for a projective plane, we glue its segment (or a topologically equivalent circle) to a Möbius strip. We then obtain a closed surface topologically equivalent to the projective plane.

5. Projective Coordinates and Projective Transformations

Investigating Euclidean space, we first introduce rectangular Cartesian coordinates, and then affine coordinates expressed in terms of the former according to

$$\begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}z + a_1 \\ y' &= a_{21}x + a_{22}y + a_{23}z + a_2 \\ z' &= a_{31}x + a_{32}y + a_{33}z + a_3 \end{aligned}$$

with a non-zero determinant of the matrix (a_{ij}). Similarly, proceeding from homogeneous coordinates x_i for a projective space, we introduce *projective coordinates* x'_i by the equations

$$\begin{aligned} x'_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 \\ x'_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 \\ x'_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 \\ x'_4 &= a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 \end{aligned} \qquad (*)$$

with a non-zero determinant of the matrix (a_{ij}). Notice that, as well as the homogeneous coordinates of a point, those projective are not simultaneously zeros, for if all x'_i vanish, then (*) have only the zero solution for x_i, the determinant being different from zero. Since homogeneous coordinates are not determined uniquely, those projective are not unique either. Viz., if x'_i are the projective coordinates of a point, then $\rho x'_i$ are also those of the same point for $\rho \neq 0$.

It is obvious that, *with respect to projective coordinates, a plane is given by a linear equation, whereas a straight line by two independent linear equations.* In fact, the equation of a plane is linear with respect to homogeneous coordinates. If we express x_i in terms of x'_i from the formulas (*), and substitute the result in the plane equation, then we obtain a linear equation for x'_i.

The four planes specified by the equations $x'_i = 0$ with respect to projective coordinates are said to be *coordinate planes*. The tetrahedron whose faces are in these planes is also said to be *coordinate*. Its vertices are

$$(1,\ 0,\ 0,\ 0),\quad (0,\ 1,\ 0,\ 0),\quad (0,\ 0,\ 1,\ 0),\quad (0,\ 0,\ 0,\ 1).$$

The point with projective coordinates (1, 1, 1, 1) is said to be *unit*.

We show that *any four planes not passing through one point can be taken to be coordinate, and each point not in any of them as unit.*

In fact, let

$$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4}x_4 = 0, \quad i = 1, 2, 3, 4$$

be the plane equations.

We introduce new coordinates x_i' by the formulas

$$x_i' = \lambda_i (a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4}x_4), \quad i = 1, 2, 3, 4.$$

In the new coordinate system, the given planes are coordinate, since $x_i' = 0$. In the new coordinate system, we can make a given point (x_1, x_2, x_3, x_4) unit (i.e., so that $x_i' = 1$) by choosing the factors λ_i.

It is obvious that the passage from one projective coordinate system to another is of the form (*). Indeed, if the equations (*) specifying the passage from homogeneous coordinates x_i to projective x_i' are solved for x_i, and the obtained expressions substituted in the formulas for the passage from x_i to projective coordinates x_i'', then we obtain formulas for the passage from x_i' to x_i'' of the form (*).

The formulas (*) can be interpreted as specifying a transformation of space, under which a point (x_1, x_2, x_3, x_4) is carried into a point (x_1', x_2', x_3', x_4') with respect to the same projective coordinate system. This transformation is said to be *projective*. It is evident that the inverse of a projective transformation is also projective. Two projective transformations performed one after the other yield a projective transformation. The identity transformation is projective. In short, *projective transformations form a group*. It is obvious that *a projective transformation sends planes into planes, and straight lines into straight lines*. As in the case of space, three projective coordinates expressed in terms of homogeneous are introduced on the plane by the formulas

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned} \qquad (**)$$

with a non-zero determinant of the matrix (a_{ij}).

With respect to projective coordinates for the plane, any straight line is given by a homogeneous linear equation

$$a_1x_1 + a_2x_2 + a_3x_3 = 0.$$

Instead of coordinate tetrahedron, the concept of *coordinate triangle* is introduced on the plane.

A transformation of a plane with respect to the same projective coordinate system, given by the formulas (**), is said to be *projective*. It is obvious that *a projective transformation carries planes into planes, and straight lines into straight lines.*

For brevity, the equations (*) specifying coordinate and projective transformations will be written from now on as $x' = Ax$, and the plane equation $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$ as $ax = 0$. Besides, we will use the superscripts x^1, x^2, x^3, x^4 for coordinates, and not x_1, x_2, x_3, x_4.

6. Cross Ratio

Let $P_1\ (x_1^i)$, $P_2\ (x_2^i)$, $P_3\ (x_3^i)$ and $P_4\ (x_4^i)$ be points in a straight line.

The number calculated by the formula

$$(P_1P_2P_3P_4) = \frac{\begin{vmatrix} x_1^i & x_1^j \\ x_3^i & x_3^j \end{vmatrix}}{\begin{vmatrix} x_1^i & x_1^j \\ x_4^i & x_4^j \end{vmatrix}} \div \frac{\begin{vmatrix} x_2^i & x_2^j \\ x_3^i & x_3^j \end{vmatrix}}{\begin{vmatrix} x_2^i & x_2^j \\ x_4^i & x_4^j \end{vmatrix}} \quad (i \neq j)$$

in terms of projective coordinates is called the *cross*, or *double*, *ratio* of the points taken in given order.

If we want the above definition to be correct, then we have to require that the value of any cross ratio should not depend on the superscripts i, j of the coordinates, in whose terms it is found.

Let

$$ax = 0, \quad bx = 0 \qquad (*)$$

be the equations of the straight line with the points P_i whose coordinates are solutions to the homogeneous system (*) of rank 2. Therefore, any of its solutions can be represented as a linear combination of two independent ones. It follows that the coordinates of P_3 and P_4 are representable in terms of those of P_1 and P_2 by the equations

$$x_3^i = x_1^i + \lambda x_2^i, \quad x_4^i = x_1^i + \mu x_2^i,$$

substituting which in the cross ratio formula, we obtain

$$(P_1P_2P_3P_4) = \frac{\lambda}{\mu}.$$

Therefore, in fact, *cross ratio is independent of the choice of coordinate superscripts i and j.*

We prove that *cross ratio is independent of the choice of a coordinate system.* Indeed, let the passage to a new coordinate system be carried out by the formula

$$x' = Ax.$$

Then

$$x_1' = Ax_1, \quad x_2' = Ax_2$$
$$x_3' = A(x_1 + \lambda x_2) = Ax_1 + \lambda Ax_2 = x_1' + \lambda x_2'.$$
$$x_4' = A(x_1 + \mu x_2) = Ax_1 + \mu Ax_2 = x_1' + \mu x_2'$$

We see that the coordinates of P_3 and P_4 are expressed in terms of those of P_1 and P_2 with respect to the new coordinate system by the same formulas as with respect to the old. Therefore, the cross ratio of the points with respect to the new coordinate system is the same, viz., λ/μ. Thus cross ratio does not depend on the choice of a coordinate system.

Cross ratio is unaltered under a projective transformation, which means that if P_1, P_2, P_3 and P_4 are sent into four points Q_1, Q_2, Q_3 and Q_4, respectively, then

$$(P_1P_2P_3P_4) = (Q_1Q_2Q_3Q_4).$$

Formally, proof is identical to the above of the cross ratio independence from the choice of a coordinate system.

Cross ratio does not change in projecting, which means that if four points in a straight line are projected from a certain point S onto another line, then they have the same cross ratio. Indeed, take S as the vertex of the coordinate triangle (0, 0, 1), and the straight line onto which the points are projected as the coordinate line $x^3 = 0$. Let $a_1x^1 + a_2x^2 + a_3x^3 = 0$ be the equation of the straight line with the points in question. The projective transformation given by

$$x^{1\prime} = x^1, \quad x^{2\prime} = x^2, \quad x^{3\prime} = a_1x^1 + a_2x^2 + a_3x^3$$

preserves the straight lines passing through S, and, therefore, sends the given points into their projections on $x^3 = 0$. Since a projective transformation is cross-ratio preserving, cross ratio remains unaltered under a projection, too, and the statement is thus proved.

The cross ratio of four concurrent straight lines in a plane is that of four points obtained in intersecting an arbitrary straight line with the four given. Since a projection leaves cross ratio unaltered, the cross ratio of straight lines so defined does not depend on a transversal.

The *cross ratio of four planes* passing through a straight line is defined similarly. We take an arbitrary line intersecting the planes, and the cross ratio of the four intersection points as that of the planes. The cross ratio of planes so defined does not depend on the choice of a transversal.

In conclusion, we give formulas to compute the cross ratio of four points in terms of their Cartesian coordinates. If we assume that the coordinates in the cross-ratio formula are homogeneous, take $i = 1$, $j = 4$, and pass from homogeneous coordinates to Cartesian, then

$$(P_1P_2P_3P_4) = \frac{x_1 - x_3}{x_1 - x_4} \div \frac{x_2 - x_3}{x_2 - x_4}.$$

If we assume $i = 2$ or $i = 3$, then we obtain a similar formula, with x replaced by y or z, respectively.

7. Harmonic Separation of Pairs of Points

We say that two points C and D in a straight line *separate harmonically* two points A and B if $(ABCD) = -1$, with the immediate consequence that *if C and D separate harmonically A and B, then A and B do the same for C and D.*

A complete *quadrangle* is a set of four points in a plane, each three of them non-collinear, together with the six straight lines joining them. The points are termed *vertices*, and the straight lines joining them *edges*. The edges of a complete quadrangle without common vertices are said to be *opposite*. The points where the opposite edges meet are said to be *diagonal*. In Fig. 120, the vertices of the complete quadrangle are P, Q, R and S, whereas the diagonal points A, B and T.

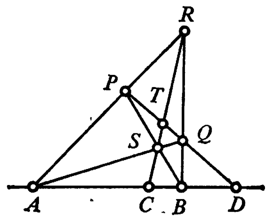

Fig. 120

Let A and B be two diagonal points of the complete quadrangle $PQRS$, and C and D those where the straight line AB cuts the edges concurrent at the third diagonal point. Then C and D separate harmonically A and B.

Proof. Let A, B and R be the vertices of the coordinate triangle, and S the unit point, viz.,

$$A\ (1,\ 0,\ 0), \quad B\ (0,\ 1,\ 0), \quad R\ (0,\ 0,\ 1), \quad S\ (1,\ 1,\ 1).$$

To find the coordinates of C, we remember that they are expressed in terms of those of S and R. We have $x^1 = 1 + \lambda \cdot 0$, $x^2 = 1 + \lambda \cdot 0$, $x^3 = 1 + \lambda \cdot 1$. Since $x^3 = 0$ on the straight line AB, the coordinates of C are 1, 1, 0. The coordinates 1, 0, 1 and 0, 1, 1 of P and Q are found similarly. Those of D are expressed in terms of the coordinates of P and Q, viz., $x^1 = 1 + \lambda \cdot 0$, $x^2 = 0 + \lambda \cdot 1$, $x^3 = 1 + \lambda \cdot 1$. Since $x^3 = 0$, $\lambda = -1$. Therefore, the coordinates of D are 1, -1, 0. Knowing those of A, B, C and D, we find easily that $(ABCD) = -1$.

A, B, C and D are projected from R into P, Q, T and D. Hence, P and Q separate harmonically T and D. P, Q, T and D are projected from A into R, S, T and C. Therefore, R and S separate harmonically T and C.

We now clarify what is the mutual disposition of A, B and C in a Euclidean straight line if D is at infinity. Take AB as the x-axis. Let D approach infinity, remaining finite.

We have

$$(ABCD)=\frac{x_1-x_3}{x_1-x_4}\div\frac{x_2-x_3}{x_2-x_4}=\frac{x_1-x_3}{x_2-x_3}\div\frac{x_1-x_4}{x_2-x_4}=-1.$$

As $x_4\to\infty$, the ratio $(x_1-x_4)/(x_2-x_4)\to 1$. Therefore, $(x_1-x_3)/(x_2-x_3)\to -1$, i.e., C approaches the mid-point of AB without limit. When D is at infinity, i.e., the straight lines AB and PQ are parallel, C will be the mid-point of the segment AB.

This property permits us to solve the following problem of elementary geometry.

Given a line segment AB and its mid-point C. Only by means of a ruler, draw a straight line PQ parallel to the straight line AB through an arbitrary point P.

Solution. Draw the straight line AP. Take any point R in it, other than A and P. Draw the straight lines RC, PB, and find a point S where they meet. Draw the straight line AS until it meets the straight line RB at a point Q. The straight line PQ is parallel to AB.

The following problem is solved similarly.

Given two parallel straight lines and a line segment on one of them. Using only a ruler, bisect the line segment (i.e., find its mid-point).

8. Curves of the Second Degree and Quadric Surfaces

A locus of points in a plane, satisfying an equation of the form

$$a_{11}x^{1^2}+2a_{12}x^1x^2+\ldots+a_{33}x^{3^2}=0 \qquad (*)$$

is called a *curve of the second degree* (or *conic*). The definition is invariant with respect to the choice of a projective coordinate system, since the passage to another one is related to a linear transformation of the variables, and, therefore, does not affect the form of the equation.

It is known from algebra that the quadratic form on the left-hand side of (*) can be reduced to one of the canonical forms

$$x^{1^2}+x^{2^2}+x^{3^2},\quad x^{1^2}+x^{2^2}-x^{3^2},\quad x^{1^2}+x^{2^2},$$
$$x^{1^2}-x^{2^2},\quad x^{1^2}$$

by a linear transformation.

From the projective geometry standpoint, this algebraic result can be interpreted as the existence of a projective coordinate system, with respect to which the equation of the given second-degree curve takes one of the forms

$$x^{1^2}+x^{2^2}+x^{3^2}=0,\ x^{1^2}+x^{2^2}-x^{3^2}=0,$$

$$x^{1^2}-x^{2^2}=0,\ x^{1^2}+x^{2^2}=0,\ x^{1^2}=0. \qquad (**)$$

In the first case, the curve is said to be *imaginary*. No point in the plane satisfies it, since projective coordinates cannot be zero simultaneously. In the second case, the curve is called an *oval*. In the third case, it splits into two straight lines $x^1-x^2=0$, $x^1+x^2=0$, and in the fourth, into two imaginary lines $x^1-ix^2=0$, $x^1+ix^2=0$. Finally, the curve decomposes into two coincident lines $x^1=0$.

The algebraic result that the left member of the equation (*) is reducible to canonical form can be interpreted differently, viz., as the possibility to convert by a projective transformation the given curve (*) into one of those given by (**) with respect to the same projective coordinate system.

Quadric surfaces are defined similarly, viz., as loci of points in space, satisfying an equation of the form

$$\sum_{i,\,j=1}^{4} a_{ij}x^ix^j=0$$

with respect to projective coordinates x^i. The existence of a projective coordinate system, with respect to which the surface equation takes one of the canonical forms

$$x^{1^2}+x^{2^2}+x^{3^2}+x^{4^2}=0,\qquad x^{1^2}+x^{2^2}+x^{3^2}-x^{4^2}=0,$$

$$x^{1^2}+x^{2^2}-x^{3^2}-x^{4^2}=0,\qquad x^{1^2}+x^{2^2}+x^{3^2}=0,$$

$$x^{1^2}+x^{2^2}-x^{3^2}=0,\ x^{1^2}+x^{2^2}=0,\qquad x^{1^2}-x^{2^2}=0,\ x^{1^2}=0$$

is proved as for curves of the second degree. This can be also regarded as the possibility to reduce a given surface by a projective transformation into that given by one of the above equations.

The *tangent* to a curve of the second degree at a point $A_0\,(x_0^i)$ is the limiting position of the secant passing through A_0 and a point of the curve $\bar{A}\,(\bar{x}^i)$ near to it as $\bar{A}\to A_0$, i.e., as $\bar{x}^i\to x_0^i$, $i=1, 2, 3$.

Make up the equation of a tangent to a curve of the second degree, given by the equation $\sum a_{ij}x^ix^j=0$. Let ω be a small neighbourhood of A_0, and $A\,(x^i)$ an outside point in the secant $A_0\bar{A}$. We normalize its coordinates, so that $\sum (x^i)^2=1$. Those of $\bar{A}$ can be expressed in terms of A_0 and A. Viz., $\bar{x}^i=x_0^i+\lambda x^i$, substituting which in the curve equation, we get

$$\lambda^2\sum a_{ij}x^ix^j+2\lambda\sum a_{ij}x^ix_0^j+\sum a_{ij}x_0^ix_0^j=0.$$

The third addend on the left-hand side is zero, since A_0 is on the curve. Cancelling λ, and passing to the limit as $\bar{A} \to A_0$, or as $\lambda \to 0$, we obtain an equation which the limiting line, i.e., the tangent, does satisfy, viz.,

$$\sum a_{ij} x^i x_0^j = 0.$$

Remark. We considered the neighbourhood ω and normalization of coordinates x^i in order to conclude from $\bar{A} \to A_0$ that $\lambda \to 0$, and $\lambda \sum a_{ij} x^i x^j \to 0$.

The locus of plane section tangents passing through a given point is called the *tangent plane* to the surface at the point. The derivation of the equation for a plane tangent to a quadric surface is in no way different from that of a tangent to a curve of the second degree, and we obtain the following tangent plane equation, viz.,

$$\sum_{i,\, j=1}^{4} a_{ij} x^i x_0^j = 0.$$

9. Steiner Theorem

The totality of all straight lines passing through one point is called a *pencil of lines*. The point is termed its *centre*. The correspondence between the lines of two pencils is called *projective* if there is a projective transformation sending the lines of one into the corresponding lines of the other. If the corresponding lines of two pencils intersect on one straight line, then such a correspondence is called a *perspectivity*. Obviously, it is a projective transformation.

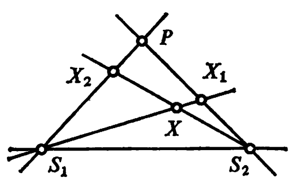

Fig. 121

The following *Steiner theorem* is valid.

The locus of points where the corresponding straight lines of two projective, but not perspective, pencils meet, is a non-singular conic. Conversely, two pencils with centres on a curve of the second degree and the corresponding straight lines intersecting on the curve, are projective.

Proof. Take the centres of the pencils as those of the coordinate triangles S_1 (1, 0, 0), S_2 (0, 1, 0), and any point P (0, 0, 1) as the third centre (Fig. 121). Let X (x^i) be the point where the corresponding straight lines of the pencils meet. Find the coordinates of the point X_1 (x_1^i) where the straight line S_1X cuts the straight line S_2P, expressing them in terms of those of S_1 and X as $x_1^1 = x^1 + \lambda \cdot 1$, $x_1^2 = x^2 + \lambda \cdot 0$, $x_1^3 = x^3 + \lambda \cdot 0$. Similarly, we find the coordinates of the point X_2 (x_2^i) where the straight lines S_2X and S_1P intersect, viz., $x_2^1 = x^2$, $x_2^3 = x^3$.

Now, let A_i (a_i^j), $i = 1, 2, 3$, be the points where the straight lines of the pencil with centre S_1 meet S_2P, and let B_i (b_i^j), $i = 1, 2, 3$, be the points where the corresponding straight lines of the second pencil meet S_1P. Since the pencils are projective, $(A_1A_2A_3X_1) = (B_1B_2B_3X_2)$, and the locus equation is

$$\frac{\begin{vmatrix} a_1^2 & a_1^3 \\ a_3^2 & a_3^3 \end{vmatrix}}{\begin{vmatrix} a_1^2 & a_1^3 \\ x^2 & x^3 \end{vmatrix}} \div \frac{\begin{vmatrix} a_2^2 & a_2^3 \\ a_3^2 & a_3^3 \end{vmatrix}}{\begin{vmatrix} a_2^2 & a_2^3 \\ x^2 & x^3 \end{vmatrix}} = \frac{\begin{vmatrix} b_1^1 & b_1^3 \\ b_3^1 & b_3^3 \end{vmatrix}}{\begin{vmatrix} b_1^1 & b_1^3 \\ x^1 & x^3 \end{vmatrix}} \div \frac{\begin{vmatrix} b_2^1 & b_2^3 \\ b_3^1 & b_3^2 \end{vmatrix}}{\begin{vmatrix} b_2^1 & b_2^3 \\ x^1 & x^3 \end{vmatrix}}.$$

For the coordinates of X, we obtain a homogeneous equation of the second degree. Therefore, the locus of X is a curve of the second degree.

We now prove the second statement of the theorem.

Let S_1, A_1, A_2, A_3 and S_2 be five points on a non-singular conic. There exists a projective transformation sending S_1, A_1, A_2, A_3 into S_2, A_1, A_2, A_3, and establishing a projectivity between the pencils with vertices S_1 and S_2. The corresponding straight lines intersect on a certain curve of the second degree (proved), S_1, A_1, A_2, A_3, S_2 belonging to the curve. To complete the proof, we have to show that two non-singular conics with five points in common are coincident.

Carry out a projective transformation under which one curve is given by the equation $y = x^2$ with respect to Cartesian coordinates and the other by an equation of general form $F(x, y) = 0$. Substituting $y = x^2$ in the second, we obtain a fourth-degree polynomial $F(x, x^2) = 0$. Since the curves have five points in common, the polynomial is equal to zero for five values of x, and, therefore, is zero identically. Hence, $y = x^2$ is wholly on the curve $F(x, y) = 0$. Interchanging them, we conclude that the second curve lies on the first, and they are coincident.

10. Pascal Theorem

We now prove the following *Pascal theorem*.

Let γ *be a non-singular curve of the second degree*, and $A_1, A_2, \ldots, A_6$ *six points in it. Then the three points where the straight lines* A_1A_5 *and* A_2A_4, A_3A_4 *and* A_1A_6, A_2A_6 *and* A_3A_5 *meet are in the same straight line* (Fig. 122).

Usually, the Pascal theorem is stated rather simply, viz., *the opposite sides of a hexagon inscribed in a curve of the second degree intersect on one straight line, understanding by the hexagon any closed broken line of six segments, and by its sides the straight lines containing the segments.*

Obviously, it suffices to prove the Pascal theorem for any non-singular conic, since any two non-singular conics are reduced into each other by a projective transformation, while any projective transformation sends straight lines into straight lines, and, in particular, points in a straight line into those in another straight line.

Let γ be the parabola $x = y^2$, and $\alpha_{ij}(x, y) = 0$ the equation of a straight line A_iA_j. We form the expression

$$P(x, y) = \alpha_{24}\alpha_{16}\alpha_{35} - \lambda\alpha_{34}\alpha_{26}\alpha_{15} \qquad (*)$$

which is a polynomial of the third degree with respect to x, y. The coordinates of $A_1, A_2, \ldots, A_6$ satisfy the equation $P(x, y) = 0$, for the first and second addends of $P(x, y)$ vanish.

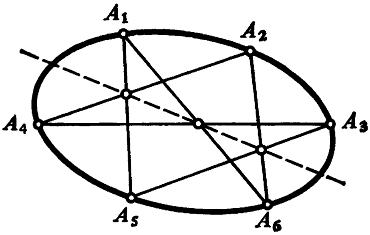

Fig. 122

Take any point A in γ, other than A_i, and choose λ, so that the coordinates of A also satisfy $P(x, y) = 0$. Seven points of γ will then satisfy $P(x, y) = 0$.

If we substitute y^2 for x in $P(x, y) = 0$, then we obtain the equation $P(y^2, y) = 0$ of the sixth degree, but satisfied by seven different values of y, or seven points. It is known that it must be an identity, and, therefore, satisfied by any y, which means that each point on the parabola γ satisfies $P(x, y) = 0$.

Regarding $P(x, y)$ as a polynomial in x with coefficients as those in y, we divide it by $x - y^2$, and obtain

$$P(x, y) = (x - y^2)\,Q(x, y) + R(y),$$

where $Q(x, y)$ is the quotient, and $R(y)$ the remainder polynomials.

Since each point on the parabola $x - y^2 = 0$ satisfies $P(x, y) = 0$, $R(y)$ is zero for all y, i.e., $R(y) \equiv 0$. Thus,

$$P(x, y) = (x - y^2)\,Q(x, y),$$

where $Q(x, y)$ is a polynomial. Since $P(x, y)$ is a polynomial of the third degree, $Q(x, y)$ is that of the first degree, and

$$P(x, y) = (x - y^2)(ax + by + c).$$

It follows from (*) that the points mentioned in the statement of the Pascal theorem satisfy $P(x, y) = 0$. Since they are not on γ, i.e., the parabola $x = y^2$, they belong to the straight line $ax + by + c = 0$.

Q.E.D.

We prove the *Pappus theorem* as a corollary.

Given two straight lines with three points A_1, A_2, A_3 in one of them, and another three A_4, A_5, A_6 in the other. Then the three points where the straight lines A_1A_5 and A_2A_4, A_1A_6 and A_2A_5, A_2A_6 and A_3A_5 meet are in one straight line.

Proof. Let

$$a_{11}x^2 + 2a_{12}xy + \ldots + a_{33} = 0 \qquad (**)$$

be the equation of our pair of straight lines. We can make it that of a non-singular conic

$$a'_{11}x^2 + 2a'_{12}xy + \ldots + a'_{33} = 0 \qquad (***)$$

by an arbitrarily small change of the coefficients in (**).

Mark points $B_1, B_2, \ldots, B_6$ in it, nearest to $A_1, A_2, \ldots, A_6$. By the Pascal theorem, the three points where the straight lines B_1B_5, and B_2B_4, ... meet are in one straight line. Now, let the coefficients in (***) tend to the corresponding ones in (**). Then B_i approach A_i indefinitely. Hence, the three points where A_1A_5 and A_2A_4, ... meet are in one straight line.

Q.E.D.

11. Pole and Polar

Let γ be a non-singular conic, and A_0 (x_0^i) an outside point. Draw a straight line through A_0, intersecting γ at two points which we denote by A_1 (x_1^i) and A_2 (x_2^i). Let X (x^i) be a point in this straight line, along with A_0 separating A_1 and A_2 harmonically. We show that all the points X so determined are in one straight line called the *polar* of A_0, and A_0 its *pole*.

We now find the equation of the polar.

Let

$$\sum a_{ij}x^ix^j = 0$$

be that of γ. Express the coordinates of A_1 and A_2 in terms of those of X_0 and X. We have

$$x_1^i = x^i + \lambda x_0^i, \quad x_2^i = x^i + \mu x_0^i.$$

Since $(A_1A_2A_0X) = \lambda/\mu = -1$, $\mu = -\lambda$. Because A_1 and A_2 are on γ,

$$\sum a_{ij}(x^i + \lambda x_0^i)(x^j + \lambda x_0^j) = 0,$$
$$\sum a_{ij}(x^i - \lambda x_0^i)(x^j - \lambda x_0^j) = 0.$$

Subtracting termwise, we obtain an equation

$$\sum a_{ij}x^ix_0^j = 0 \qquad (*)$$

satisfied by the coordinates of X. Being linear, it is, therefore, that of a straight line, just the polar of A_0.

If A_0 is on the curve, our construction does not make sense, and we then define the polar formally as the straight line given by (*).

It easily follows from the polar equation that *if the polar of* (x_0^i) *passes through* (x_1^i), *then that of* (x_1^i) *passes through* (x_0^i).

In fact, the polar of (x_0^i) has the equation

$$\sum a_{ij}x^i x_0^j = 0,$$

whereas that of (x_1^i)

$$\sum a_{ij}x^i x_1^j = 0.$$

If the polar of (x_0^i) passes through (x_1^i), then

$$\sum a_{ij}x_1^i x_0^j = 0.$$

And if $a_{ij} = a_{ji}$, then

$$\sum a_{ij}x_0^i x_1^j = 0,$$

i.e., the polar of (x_1^i) passes through (x_0^i), thus completing the proof.

Hence, *if a point moves along a straight line, then its polar always*

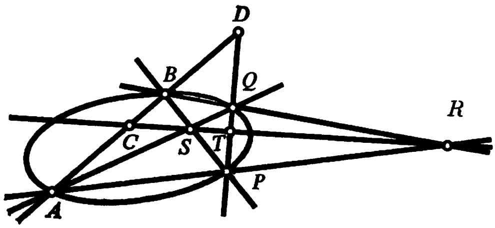

Fig. 123

passes through the pole of this line. Conversely, if a straight line passes through a given point and rotates, then its pole moves along the polar of the point.

Two straight lines are said to be *conjugate* if each of them passes through the pole of the other. The conjugate diameters of a central curve of the second degree are conjugate. The pole of each is the point at infinity of the other. (Recall that the point half-way between two points in a straight line is the harmonic conjugate of the point at infinity.)

The polar of a point admits a simple geometric construction (Fig. 123). Viz., draw through a given point D two straight lines, each intersecting the curve at two points. The polar of D passes through the diagonal points R and S of the complete quadrangle $ABPQ$. In fact, by the property of a complete quadrangle, C, D separate harmonically A, B, whereas D, T the points P, Q. Therefore, C and T are on the polar of D.

The solution of the following elementary geometry problem is based just on the property of a pole and polar.

Given a circle and an outside point, construct the tangents from the point to the circle by means of a ruler only.

Solution (Fig. 124). Construct the polar a of A. The points where it meets the circle are the points of tangency. In fact, the tangents are the polars of the points of contact, and, therefore, pass through the pole of a, or A.

We now turn to the Klein model of Lobachevskian geometry, and clarify how perpendicular straight lines are represented there. If

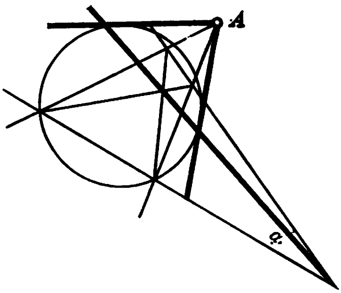

Fig. 124

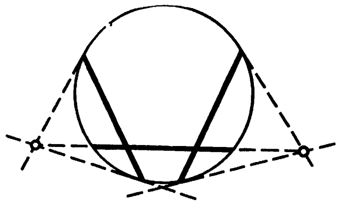

Fig. 125

two straight lines intersect at the centre of a circle, then to be perpendicular in the sense of Lobachevsky means the usual perpendicularity according to Euclid. Perpendicular diameters are conjugate. Since Lobachevsky motions in the Klein model are circle-preserving projective transformations, to be perpendicular for two straight lines in general position is to be conjugate with respect to the circumference of the Klein circle. Thus, those straight lines, or chords of the circle, are perpendicular to a given straight line, which pass through the pole.

It should be noted in connection with the above that one, and one only, perpendicular can be drawn to two non-intersecting and non-parallel (in the sense of Lobachevsky) straight lines in the Lobachevsky plane. (How such a perpendicular is constructed in the Klein model is shown in Fig. 125.)

12. Polar Reciprocation. Brianchon Theorem

Let γ be a non-degenerate conic. Map the set of points and straight lines in a projective plane onto itself, associating an arbitrary point with its polar relative to γ, and an arbitrary straight line with its pole. We call this mapping *polar reciprocation.*

Polar reciprocation possesses an important property following from those of a pole and the polar. Viz., if two points A and B are associated with two straight lines a and b, then the line AB is associated with the point where they meet. If a and b are associated with A and B, then the point of their intersection is associated with the straight line AB.

Apply polar reciprocation to the proof of the following *Brianchon theorem.*

The straight lines joining the opposite vertices of a hexagon circumscribed about a non-degenerate conic are concurrent (Fig. 126).

By the hexagon, we understand any closed broken line of six segments, and by its sides the straight lines containing the segments.

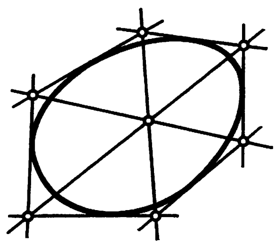

Fig. 126

Polar reciprocation with respect to the curve with the circumscribed hexagon sends the sides into their points of tangency with the curve, while the vertices into the straight lines joining the corresponding points of contact. We obtain an inscribed hexagon. By the Pascal theorem, its opposite sides intersect on a straight line associated with the point through which the straight lines pass, joining the opposite vertices of the circumscribed hexagon. Q.E.D.

The concept of the polar of a point relative to a non-degenerate quadric surface is introduced similarly to a plane. Here, however, the polar is a plane.

If a surface is given by an equation

$$\sum_{i,\, j=1}^{4} a_{ij} x^i x^j = 0,$$

then the polar of the point (x_0^i) by

$$\sum a_{ij} x^i x_0^j = 0.$$

The concept of polarity in *space* is introduced in terms of a pole and the polar. This is a mapping of the set of points, straight lines and planes in space, under which an arbitrary point is associated with its polar, an arbitrary plane with its pole, and an arbitrary straight line with that on which the polars of any two of its points meet.

13. Duality Principle

We now dwell on one of the basic facts of projective geometry, viz., the duality principle.

If, in the axioms of incidence, we replace the expression "a point is in a straight line" by "a point is incident with a straight line", and "a straight line passes through a point" by "a straight line is incident with a point", then, on replacing in each axiom the term "point" by "straight line", and "straight line" by "point", we obtain statements which hold due to the corresponding axioms.

In fact, the new version of Axiom I_1 states that, for two points A and B, there exists a straight line incident with them. The corresponding statement that, for two straight lines, there exists a point incident with them follows from Axiom I_2.

Axiom I_2. For two distinct points A and B, there is not more than one straight line incident with them. The corresponding statement follows. Viz., for two distinct straight lines a and b, there exists not more than one point incident with them.

Axiom I_3. For a given straight line, there exist three points incident with it. There are three points not incident with one straight line.

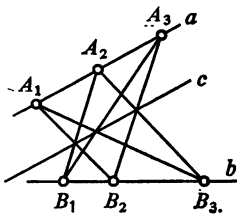

Fig. 127

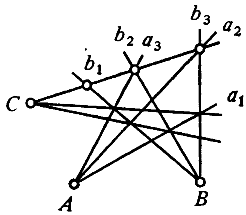

Fig. 128

The corresponding statement is that, for a given point A, there exist three straight lines incident with it, and that there are three straight lines not incident with one point. In fact, due to Axiom I_3, there exist two points B and C not in one straight line with A. By the same axiom, there are three points on the straight line BC. The straight lines in question join the three points to A. The second statement also follows from Axiom I_3. In fact, join three non-collinear points pairwise. The three straight lines obtained are not concurrent.

In the independent construction of plane projective geometry, i.e., not in space, D. Hilbert showed that the Desargues theorem should be regarded as an axiom of incidence. However, its self-duality is obvious.

Duality also turns out to occur in the other axioms for plane projective geometry, and not only in the axioms of incidence. We then have the *principle of duality* for the plane.

If a certain statement A holds for points and straight lines, and is expressed in terms of incidence and order, then a statement A′ is also valid, when the term "point" is replaced by "straight line", and "straight line" by "point".

E.g., let three points A_1, A_2 and A_3 be incident with a straight line a, B_1, B_2 and B_3 three points incident with a straight line b, and C_{ij}, $i \neq j$, the points incident with the straight lines A_iB_j and A_jB_i. Then C_{ij} are incident with one straight line c (Fig. 127). This is the *Pappus theorem.*

Now, the dual statement. Let three straight lines a_1, a_2 and a_3 be incident with a point A, b_1, b_2 and b_3 three straight lines incident with a point B, and c_{ij}, $i \neq j$, the straight lines incident with the points a_ib_j and a_jb_i, etc. Then c_{ij} are incident with one point C (Fig. 128).

The duality principle is also valid in projective space, i.e., the validity of any proposition A for points, straight lines and planes entails a statement A', where the term "point" is replaced by "plane", and "plane" by "point".

In projective geometry, duality naturally receives analytic expression. Below, we illustrate this fact.

We call the coefficients of the equation of a straight line its *tangential coordinates*, (as well as those of a point) obviously defined only up to an arbitrary nonzero multiplier.

For fixed u_1, u_2 and u_3, the equation

$$u_1x_1 + u_2x_2 + u_3x_3 = 0$$

is known to be that of a straight line with u_1, u_2, u_3 as coordinates, and that of a pencil of straight lines with centre (x_1, x_2, x_3) for fixed x_1, x_2 and x_3.

It is also known that, for any two points (y_i) and (z_i) on a straight line, the coordinates of an arbitrary point on it can be represented in the form $x_i = \lambda y_i + \mu z_i$. Similarly, for any two straight lines (v_i) and (w_i) of a pencil, the coordinates of an arbitrary straight line in it can be represented as $u_i = \lambda v_i + \mu w_i$.

Finally, we can show that the cross ratio of four straight lines in a pencil is determined by the same formula, with the coordinates of points replaced by the coordinates of the lines.

In space, the tangential coordinates of planes are introduced similarly, and analogous facts established.

A *curve of the second class* is a figure formed by all straight lines whose coordinates satisfy an equation

$$b_{11}u_1^2 + 2b_{12}u_1u_2 + \ldots + b_{33}u_3^2 = 0.$$

It is formed either by the tangents to a curve of the second degree or consists of two pencils of possibly coinciding straight lines.

14. Various Geometries in Projective Outlook

In his work *Vorlesungen über nicht-Euklidische Geometrie*, F. Klein has established a remarkable relation between Euclidean, Lobachevskian and Riemannian geometries, the latter in the narrow sense.

In Ch. XV, we considered a model of Lobachevskian geometry on the Euclidean plane in the circle $x^2 + y^2 < 1$. It is evident that this model can be regarded as valid for a projective plane in the domain $x_1^2 + x_2^2 - x_3^2 < 0$ bounded by a curve of the second degree.

The question arises, can a similar model be obtained on a projective plane for Euclidean geometry, too?

It is easily seen not to be very hard to do, and that the Cartesian model considered in Ch. XV is the one. In fact, we call the points on a projective plane with $x_3 \neq 0$ *points of a Euclidean plane*, and projective transformations of the form

$$\begin{aligned} x_1' &= x_1 \cos\theta - \varepsilon x_2 \sin\theta + a_1 x_3 \\ x_2' &= x_1 \sin\theta + \varepsilon x_2 \cos\theta + a_2 x_3, \qquad \varepsilon = \pm 1 \\ x_3' &= x_3 \end{aligned} \tag{$*$}$$

we call *motions*.

If the straight line $x_3 = 0$ is said to be *at infinity*, and Cartesian coordinates are referred to, then the transformations become

$$\begin{aligned} x' &= x\cos\theta - \varepsilon y \sin\theta + a_1 \\ y' &= x \sin\theta + \varepsilon y \cos\theta + a_2 \end{aligned}.$$

In the Cartesian model of Euclidean geometry, motions are given by precisely the same formulas.

The projective transformations (*) can also be characterized geometrically. They preserve the singular curve of the second class $u_1^2 + u_2^2 = 0$. Indeed, it consists of two pencils of straight lines $u_1 + iu_2 = 0$, $u_1 - iu_2 = 0$ with centres at the points $(1, i, 0)$, $(1, -i, 0)$.

It is easy to see that (*) either leaves the points fixed ($\varepsilon = 1$) or interchanges them ($\varepsilon = -1$), thus preserving $u_1^2 + u_2^2 = 0$. It should be noted, however, that the projective transformations determined by the above geometric property do not include the transformations (*) solely, and have a more general form

$$\begin{aligned} x_1' &= \rho\,(x_1 \cos\theta - \varepsilon x_2 \sin\theta) + a_1 x_3 \\ x_2' &= \rho\,(x_1 \sin\theta + \varepsilon x_2 \cos\theta) + a_2 x_3\,. \\ x_3' &= x_3 \end{aligned}$$

containing similitudes, and not exclusively motions.

The axiom system for Riemannian geometry in the narrow sense consists of the axioms of incidence, order, continuity axiom for projective geometry and axioms of congruence for Euclidean geometry, admitting a model similar to the above. Viz., all the axioms hold on the plane if by a point we understand a point on a projective plane, by a straight line a projective line, relations of incidence and order in the sense of projective geometry, and, finally, by motions projective transformations preserving the imaginary non-degenerate conic $x_1^2 + x_2^2 + x_3^2 = 0$.

A similar model is valid in the axioms for space.

The curves of the second class $u_1^2 + u_2^2 \pm u_3^2 = 0$ are formed by the tangents to the second-degree curves $x_1^2 + x_2^2 \pm x_3^2 = 0$. Any projective transformation preserving $x_1^2 + x_2^2 \pm x_3^2 = 0$ then also preserves $u_1^2 + u_2^2 \pm u_3^2 = 0$. Hence, projective transformations preserving a curve of the second class $u_1^2 + u_2^2 + \varepsilon u_3^2 = 0$ correspond to motions in Riemannian geometry if $\varepsilon = +1$, to those in Lobachevskian geometry if $\varepsilon = -1$, and, finally, to Euclidean motions and similitudes if $\varepsilon = 0$.

A curve of the second degree or second class is called the *absolute* if it is invariant with respect to projective transformations associated with some or other geometry.

In considering the Klein model of Lobachevskian geometry, we have noted that the distance between two points A and B in the Lobachevsky plane is equal to the logarithm of the cross ratio of four points, the two given and two points where the straight line AB meets the absolute. A similar result also holds for Riemannian geometry. In all the geometries, the angle between two straight lines a and b is measured by the logarithm of the cross ratio of four straight lines, of which two are a and b, and the other two belong to the pencil ab and the absolute (as a curve of the second class).

EXERCISES TO CHAPTER XVI

1. Given that $AB \parallel A_1B_1$, $BC \parallel B_1C_1$ and $AC \parallel A_1C_1$ in two triangles ABC and $A_1B_1C_1$, prove that the straight lines AA_1, BB_1, CC_1 are either concurrent or parallel.

2. Given that $AA_1 \parallel BB_1 \parallel CC_1$, $AB \parallel A_1B_1$ and $AC \parallel A_1C_1$ in two triangles ABC and $A_1B_1C_1$, prove that $BC \parallel B_1C_1$.

3. Find the homogeneous coordinates of the point at infinity on a straight line $\frac{x-a}{k} = \frac{y-b}{l} = \frac{z-c}{m}$.

4. Given that three points (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4) and (c_1, c_2, x_3, x_4) are in the same straight line, find x_3 and x_4.

5. Given three points on a straight line, prove that there exists a projective transformation sending them into $(-1, 0, 0, 1)$, $(0, 0, 0, 1)$, $(1, 0, 0, 1)$.

6. Given cross ratio $(ABCD) = \xi$, find that of the same points taken in any other order, e.g., $(CBAD)$.

7. Find the cross ratio of four straight lines $y = x \tan \alpha$, $y = x \tan \beta$, $y = x \tan \gamma$, $y = x \tan \delta$.

8. Find the Euler characteristic of a projective plane.

9. Account for the following method of constructing an ellipse (Fig. 129). Divide two line segments AC and CD into an equal number of parts, and join to B and A the corresponding division points, starting from A and C, the intersection point lying on the arc AE of the ellipse with semi-axes OA and OE.

10. We know how to construct the polar of a point with respect to a given curve of the second degree. Now, how can we find a pole if the polar is given?

11. How will the general equation to a non-degenerate conic be simplified if the straight line $x_3 = 0$ is the polar of the point (0, 0, 1)?

12. How will the equation of a non-degenerate conic be simplified if the coordinate triangle vertices are the poles of its opposite sides?

13. State the proposition dual to the Steiner theorem.

14. Show that polar reciprocation with respect to a sphere carries a regular polyhedron with centre at that of the sphere into a regular

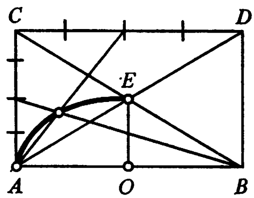

Fig. 129'

polyhedron, too, viz., a tetrahedron into a tetrahedron, a cube into an octahedron, an octahedron into a cube, a dodecahedron into an icosahedron, and an icosahedron into a dodecahedron.

15. A perpendicular PQ is dropped to a straight line a from an outside point P on the Lobachevsky plane, and a parallel straight line b (in the sense of Lobachevsky) drawn. Find the dependence of the angle between PQ and b (*parallel angle*) on the distance from P to a.

16. Prove that Lobachevsky parallel lines approach each other indefinitely in the parallelism direction, and diverge without limit in the opposite direction.

Part Four

CERTAIN PROBLEMS OF ELEMENTARY GEOMETRY

Chapter XVII

METHODS FOR SOLUTION OF CONSTRUCTION PROBLEMS

1. Preliminaries

To pose a construction problem means to require the construction of a geometric figure by means of some prescribed drawing instruments. The school course of geometry usually considers construction problems by means of compasses and ruler.

It is assumed that, by means of a ruler as an instrument for geometric constructions, we can draw an arbitrary straight line through one or two given points. No other operations can be performed; in particular, no line segments can be marked off even if the ruler has scale marks, both of its edges cannot be used, etc.

As to compasses as an instrument for geometric constructions, we can describe a circle of a given radius from a given centre. In particular, a given line segment can be cut off on a given straight line from a given point.

A solution of a given construction problem usually includes the following, viz., (i) analysis, (ii) construction, (iii) proof that the solution is correct and (iv) investigation of the solution.

The search for a solution starts with assuming that the problem has been solved, or the figure constructed. The figure is then studied (as well as its relation to the data) until the sequence of constructions leading to a solution becomes clear. To carry out the actual construction is, as a rule, not necessary, and only the proof of the solution correctness is, i.e., that we shall, in fact, obtain a figure with the required properties. By investigation, we decide whether or not the problem will always have a solution for some or other concrete data, and how many solutions it may have.

Analysis is the most difficult point. No definite recipe can be given; however, there are several methods for making it easier.

We illustrate by example.

Construct a triangle, given a side, an adjacent angle and the sum of the other two sides.

Analysis. Assume that the problem has been solved, and a triangle ABC in which $AB = c$, $\angle ABC = \theta$, $AC + BC = l$ constructed

(Fig. 130). We see that if the line segment CA is cut off on the extension of the line segment BC, then we can find the position of a point D, since $AC + BC = l$. Meanwhile, the unknown vertex C is equidistant from A and D. Hence the construction outline. Take a line segment AB equal to c, construct an angle equal to θ on the half-line

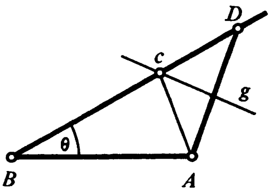

Fig. 130

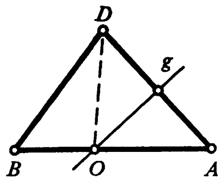

Fig. 131

BA, and mark off on the side of the angle a line segment BD equal to l, after which draw the perpendicular bisector g of AD. It meets BD in C, the vertex of the triangle.

Proof. Since g is the perpendicular bisector, $AC = CD$; therefore, $AC + BC = CD + BC = l$. Thus, $AB = c$, $\angle ABC = \theta$, $BC + AC = l$, and the constructed triangle, in fact, satisfies the conditions of the problem.

Analysis. First of all, we notice that the problem has no solution if $l \leqslant c$, since any two sides of a triangle are together greater than the third. Let $l > c$. Show that the problem has one, and only one, solution. Indeed, g intersects the side AD of the triangle ABD, and, therefore, one of the other two sides AB or BD (not passing through B, since $AB \neq BD$). If it intersected AB, we would then have $AB = AO + OB = BO + OD > BD$ (Fig. 131). But $AB < BD$, $c < l$. Consequently, g cuts BD, and the problem does have a solution. It is obvious that the solution is unique, for a straight line can intersect a line segment only at one point.

2. Locus Method

It consists in the following. Assume that we have decided during the analysis of a solution to a construction problem that it will be solved if a certain point X satisfying two conditions is found. The locus of points satisfying the first condition is a certain figure F_1, and the locus satisfying the second is a certain figure F_2. The required point X belongs both to F_1 and F_2, and is the point of their intersection.

In order that X could be found as the intersection of F_1 and F_2, it is required that the figures should admit a construction by means of our drawing instruments, i.e., compasses and a ruler, for which the figures should consist of straight lines and circles only. In this connection, of special interest are loci of points, which are straight lines and circles.

We now list some of the most important of them.

1. Locus of points equidistant from a given point is a circle with centre at the point.

2. Locus of points equidistant from a given straight line consists of two lines parallel to the given one, and at the given distance from it.

3. Locus of points equidistant from two given points is the straight line perpendicular to the line segment with ends at these points and passing through its mid-point (*perpendicular bisector*).

4. Locus of points equidistant from two given intersecting straight lines consists of the bisectors of the angles formed by the lines.

5. Locus of points, from which a line segment AB is visible at a given angle θ, and which are on one side of the straight line AB, is an arc with ends at A and B.

6. Locus of points, whose distances from two given points are in a given ratio $m : n$ ($m/n \neq 1$), is a circle (*Apollonius' circle*; see Ch. I, Sec. 4).

7. Locus of points whose distances from two given straight lines are in a given ratio λ consists of two straight lines. (If the line equations are normal,

$$ax + by + c = 0, \quad a_1x + b_1y + c_1 = 0,$$

then the straight lines of the locus are given by

$$(ax + by + c) + \lambda\,(a_1x + b_1y + c_1) = 0,$$
$$(ax + by + c) - \lambda\,(a_1x + b_1y + c_1) = 0.)$$

8. Locus of points such that the difference of squares of their distances from two given points is constant is a straight line perpendicular to that joining the points (see Ch. III, Sec. 1).

9. Locus of points such that the tangents drawn from them to two given circles are equal is a straight line if the circles are disjoint, or part of the straight line passing through the points where the circles meet, without the line segment joining the points.

We now give an example of a solution by the locus method.

Given four points A, B, C, D, find a point X such that $\angle AXB = \angle BXC = \angle CXD$.

Solution. Assume that the problem has been solved (Fig. 132). Then the line segment XB is an angle bisector in the triangle AXC. As we know, an internal bisector divides the opposite side in the ratio of the sides containing the angle bisected. Therefore,

$$AX : CX = AB : BC,$$

which means that X belongs to the locus of points, the ratio of whose distances from A and C is $AB : BC$, i.e., a circle. Similarly, we conclude that X belongs to the locus of points, also a circle, such that the ratio of the distances from B and D is $BC : CD$. The required point is where the circles meet.

3. Similarity Method

Certain problems become ill-posed if one of their conditions is dropped, admitting infinitely many solutions, and yielding figures similar to the required. If we construct one of such figures, then the required may be obtained by a similitude.

We illustrate by two examples.

Construct a triangle, given two angles and the perimeter.

Solution. If we omit the condition that the triangle should have the given perimeter, then the problem is reduced to the construction

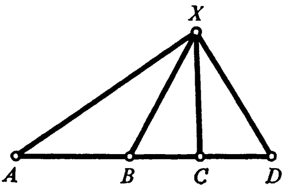

Fig. 132

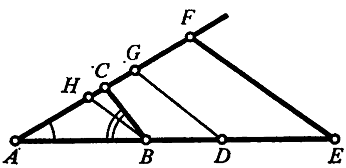

Fig. 133

of a triangle with two given angles. It is rather easy. Take an arbitrary line segment AB, and construct the given angles on the half-lines AB and BA. We obtain a triangle ABC with the given angles (Fig. 133). It is similar to the required. To obtain the desired triangle, the triangle constructed should be subjected to a suitable similitude.

Cut off on the side AB produced, line segments BD and DE equal to the sides BC and AC, and also on the half-line AC a line segment AF equal to the perimeter. Draw through B and D two straight lines parallel to EF. The line segments AH, HG and GF form the required triangle. They and the sides of the constructed triangle are proportional to the perimeters of the required and constructed triangles.

Construct a circle touching the sides of an angle, and passing through a given point.

Solution. Neglect the requirement that the circle should pass through a given point. It is easy to construct an auxiliary circle tangent to the sides of the angle, for which we make equal intercepts on the sides, and draw perpendicular straight lines through their ends. The centre of this circle is just where they meet. To obtain the required circle, the circle constructed should be subjected to a homothety with respect to the vertex of the angle, and with ratio AS/BS. The centre of the sought-for circle is at the intersection of the straight lines SO and AO' parallel to OB, where A is the point through which the circle should pass, S the vertex of the angle, B one of the points of intersection of the ray SA and the auxiliary circle, O the centre of the auxiliary, and O' that of the required circle.

4. Reflection Method

It may happen so that a figure to be constructed possesses points symmetric about a certain straight line or point, in which case it will be found useful to carry out a similitude with respect to the straight line or point, respectively.

We illustrate by two examples.

Construct a line segment AB with the given mid-point O and ends on two given straight lines a and b.

Solution. Assume that the problem has been solved. Then the ends of the segment are symmetric about O. If one of the lines, say, a, is reflected in O, then we obtain a straight line a' passing through the other end-point B. Thus, B is obtained by intersecting b with a' which is symmetric to a with respect to O. It then suffices to extend the straight line BO until it meets a, and we obtain the second end-point, A.

Given three straight lines a, b and c, construct a line segment AB perpendicular to c, with the mid-point on it, and ends on a and b.

Solution. Assume that the problem has been solved. Then the ends of the required line segment are symmetric about c. Therefore, if a is reflected in c, then it will turn into a straight line a' passing through B. Thus, B is where b meets a'. We then draw through B a straight line perpendicular to c, and thereby find the required line segment.

Note that, as well as in the previous problem, we can give here any figure instead of the straight lines a or b, and any figure admitting a construction with compasses and ruler instead of the third straight line, i.e., consisting of straight lines and circles.

5. Translation Method

It consists in translating some parts of a required figure with the purpose of obtaining a new one admitting a known construction.

We illustrate by two examples.

Construct a trapezium, given the bases and diagonals.

Solution. Assume that the problem has been solved, and a trapezium $ABCD$ constructed (Fig. 134). Translate the diagonal BD, so that its vertex B coincides with the vertex C. We now know all the sides of the triangle ACD_1, two equal to the trapezium diagonals, and the third to the sum of the bases. Hence the following solution. We first construct the triangle ACD_1, find the point D (AD being the known base of the trapezium), then draw through C a straight line parallel to AD, and another through D parallel to CD_1. They meet at B, and the trapezium $ABCD$ possesses the given bases and diagonals.

Given two circles k_1, k_2, and a straight line a. Construct a line segment $AB = d$ parallel to a with ends on k_1, k_2.

Solution. Assume that the problem has been solved, and the line segment AB constructed (Fig. 135). If one of the circles, e.g., k_1,

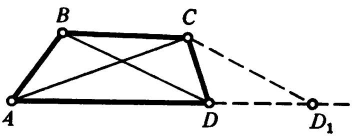

Fig. 134

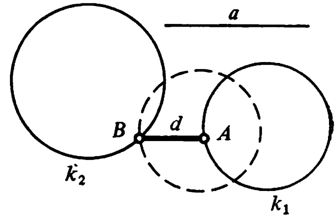

Fig. 135

is translated through d so that it remains parallel to a, then it turns into a circle passing through the other end, B. Thus, B is at the intersection of k_2 with the circle obtained by translating k_1. We then find the line segment itself by drawing through B a straight line parallel to a.

6. Rotation Method

It consists in rotating some parts of a figure with the purpose of obtaining a new one with known construction.

We illustrate by two examples.

Given two circles k_1 and k_2, and a point A. Construct an isosceles triangle with vertex A, and angle θ at A, and base vertices on k_1 and k_2.

Solution. Assume that the problem has been solved, and the triangle ABC constructed (Fig. 136). The vertex B is made coincident with the vertex C on rotating the side AC about A through θ. Hence the solution. Viz., rotate k_1 about A through θ. The obtained circle intersects k_2 at the vertex B of the required triangle. To find C, draw a circle with centre at A and radius AB until it meets k_1.

Construct a square whose sides pass through four given points A, B, C and D.

Solution. Assume that the square has been constructed (Fig. 137). Turn the line segment DB about D through 90°, and then translate it so that D coincides with A. Meanwhile, B' falls on the point B'' in the side passing through C (or in this side produced), which follows from the congruence of the right triangles BED and AFB''.

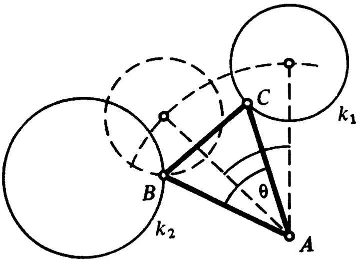

Fig. 136

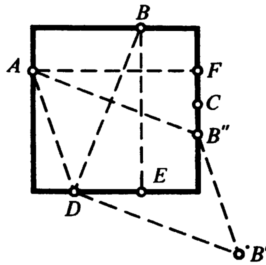

Fig. 137

We then draw the line CB'' containing this side, another line parallel to CB'' through A, and two straight lines perpendicular to the latter through B and D. The constructed square is the required.

7. Inversion Method

The inversion concept was introduced in Ch. III, Sec. 8, where we proved that a circle inverts into a circle (or a straight line if the given circle passes through the centre of inversion). A straight line which does not pass through the centre inverts into a circle, and into itself if it does.

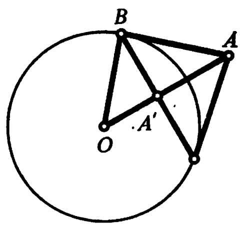

Fig. 138

Inversion can be represented geometrically as follows. Let O be its centre. Describe a circle, centre O, with radius of inversion (Fig. 138). Then an outside point A inverts into A', the intersection of OA with the chord joining the ends of the tangents from A. The proof is simple, viz., by the property of right triangles, $OA \cdot OA' = OB^2 = r^2$. A' inverts into A, and it becomes clear how to find A if A' is given.

Besides the above properties of transforming straight lines and circles, inversion possesses another remarkable one. It *preserves angles between curves*. This means that if two curves intersect at some angle, then they invert into two curves intersecting at the same angle. For proof, we notice first of all that inversion preserves tangency, i.e., if two curves touch at a certain point, or possess a common tangent, then they invert into two curves tangent at the corresponding point.

Now, let two curves γ_1 and γ_2 meet at a point A (Fig. 139). Draw through the point two tangent straight lines. A inverts into a certain point A'. Draw two circles k_1 and k_2 touching the straight lines at

A, and passing through A'. (If either γ_1 or γ_2 is tangent to OA, then we shall have just this line instead of the circle.) The circles meet at A and A' at the same angle.

Both k_1 and k_2 invert into themselves. In fact, by the property of secants, $OB \cdot OB' = OA \cdot OA' = r^2$. γ_1 and γ_2 invert into two curves

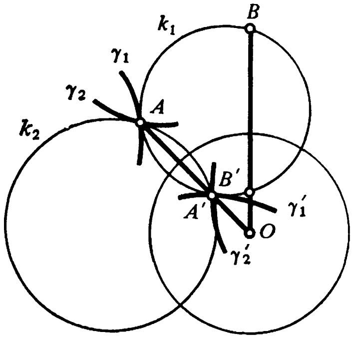

Fig. 139

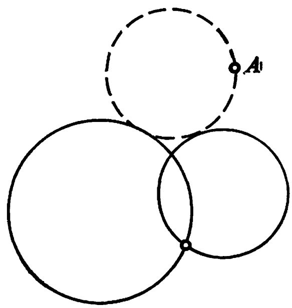

Fig. 140

γ_1' and γ_2' touching k_1 and k_2 at A'. Since the circles intersect at the same angle, γ_1' and γ_2' meet at A' at the same angle as γ_1, γ_2 at A, and the statement is thus proved.

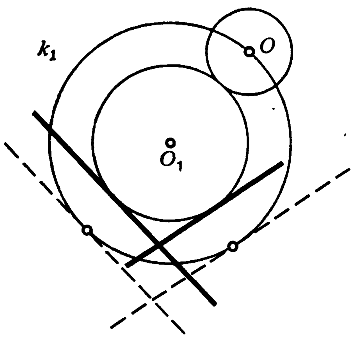

Fig. 141

We now illustrate by an example how to apply inversion to the solution of a construction problem.

Given two intersecting circles and a point A. Describe a circle passing through A, and tangent to the two given circles.

Solution. Assume that the circle has been constructed (Fig. 140). Apply inversion with respect to the point where the given circles meet. A then inverts into a certain point A', the given circles invert into straight lines, while the required circle into a circle tangent to these straight lines, passing through A', and whose construction is known (see Sec. 3). We then carry out the inverse transformation, and the constructed circle inverts into the required.

Given three circles, two of which intersecting. Construct a circle touching all the three.

Solution. As in the previous problem, we use inversion relative to the point where the two given circles meet. Two circles of the three then invert into straight lines, and the problem is reduced to the construction of a circle tangent to two straight lines and a circle (Fig. 141). A circle k_1 of radius O_1O, concentric with the required, passes through the point O, and touches two straight lines parallel to the given, which are from the latter at a distance of the radius of the given circle. Thus, the problem is again reduced to the construction of a circle passing through a given point O, and touching the two given straight lines. We then find its centre O_1 and the circle itself. Having carried out the inverse transformation, we find the required circle.

Note that inversion and the passage to concentric circles, which we have just used, permit us to solve the general problem of constructing a circle touching three given ones (*Apollonius' problem*).

8. On Solvability of Construction Problems

It is sometimes very difficult to solve a construction problem with the aid of compasses and a ruler, e.g., the *Malfatti problem* of the construction of three circles touching the sides of a triangle, and each other. There are also impossible construction problems such as duplication of the cube when it is required to find the edge of a cube with volume twice greater than that of a given one.

The answer to the question whether or not a given problem is solvable by means of compasses and ruler is supplied by the following theorem.

A problem whose analytic solution leads to an equation unsolvable by radicals is a construction which is impossible with compasses and ruler. Conversely, if the analytic solution of a problem leads to an answer involving only rational operations and taking square roots, then the construction is possible.

In fact, suppose that the construction is possible. Let the base plane be the xy-plane. Drawing straight lines and circles, and performing simultaneous computations related to the determination of intersection points, we then come to expressions involving only rational operations and taking square roots, which proves the first part of the theorem.

Conversely, if the analytic solution of a problem leads to an answer only involving rational operations and taking square roots, then the answer can be found by construction by compasses and ruler. For proof, it suffices to recall that expressions of the form $a + b$, $a - b$, $\frac{ab}{c}$, $\sqrt{ab}$, $\sqrt{a^2 + b^2}$, where a, b and c are three given line segments, can be constructed by compasses and ruler.

We illustrate by two examples.

Construct the side of a regular decagon inscribed in a circle of radius R.

Solution. The side of a regular decagon is the base of an isosceles triangle whose sides are equal to R, and the vertex angle is 36° (Fig. 142). Its bisector drawn from a base vertex separates it into two isosceles triangles AOC and ABC. Therefore, $AB = AC = OC$.

By the property of an angle bisector,

$$\frac{BC}{AB}=\frac{OC}{OA}.$$

Hence, denoting AB by x, we obtain

$$\frac{R-x}{x}=\frac{x}{R}, \quad x^2+Rx-R^2=0.$$

Its positive root is

$$x=\frac{-R+\sqrt{5R^2}}{2}=\frac{\sqrt{5R\cdot R}-R}{2}.$$

To construct a line segment of length x is easy. Take a semicircle with diameter $6R$, and drop the perpendicular SN to the

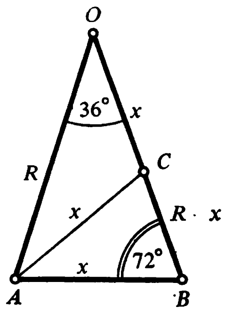

Fig. 142

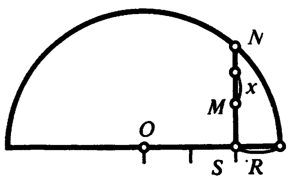

Fig. 143

diameter (Fig. 143). Cut off $SM = R$. Therefore, half the line segment MN is of the required length x.

The problem of duplication of a cube leads to the equation $x^3 - 2 = 0$, where x is the side of the cube whose volume is twice the unit one. It has been proved that the roots cannot be solvable by radicals. Therefore, the problem of duplication of a cube is unsolvable by means of compasses and ruler.

Another impossible construction problem unsolvable is that of trisecting an angle when it is required to divide an arbitrary angle into three equal parts. Analytically, it also leads to an equation of the third degree, which is generally unsolvable by radicals.

EXERCISES TO CHAPTER XVII *

1. Construct a circle of given radius, touching two given ones.
2. Find a point from which two given line segments are visible at given angles.

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* Borrowed from A. Adler's *Theory of Geometric Constructions* (Uchpedgiz, Moscow, 1940).

3. Inscribe a right triangle in a given circle, so that the sides containing the right angle pass through two given points.

4. Given three circles of the same radius, construct a circle touching them externally.

5. Find a point from which the sides of a given triangle are visible at 120°.

6. Construct a circle intersecting three given ones at right angles.

7. Find a point from which three given circles are visible at the same angle.

8. Given a straight line AC and an outside point B, find such a point X on AC that $AX + XB$ equals a given line segment l.

9. Given two concentric circles and a point P, draw a straight line through P so that the line segment contained by the circles is visible from their centre at a given angle α.

10. Given two pairs of parallel straight lines and a point P, draw a straight line through P so that both pairs make equal intercepts on it.

11. Given two circles and a point, draw a straight line through it so that the circles cut off chords of given length.

12. Inscribe a parallelogram with given directions of the sides into a given quadrilateral.

13. Construct a square, given the sum of its side and diagonal.

14. Inscribe a square in a given triangle.

15. Given a circle with two radii, construct a chord trisected by them.

16. Inscribe in a given quadrilateral a rhombus so that its sides are parallel to the diagonals of the quadrilateral.

17. Describe a circle touching a given straight line, and passing through two given points.

18. Construct a triangle, given its altitudes.

19. Construct a triangle, given an angle, the altitude and bisector drawn from the vertex of the angle.

20. Construct a triangle, given a median and altitude drawn from the same vertex and the circumradius.

21. Construct a triangle, given a side, the sum of the other two sides and the altitude drawn on one of them.

22. Construct a triangle, given a side, the opposite angle and the sum of the other two sides.

23. Draw a straight line through the point where two given circles meet so that the sum of the chords cut off is greatest (when the chords do not overlap).

24. Construct a triangle, given the perimeter, circumradius and one of the angles.

25. Construct a triangle, given the three medians.

26. Construct a parallelogram, given the diagonals and angle between them.

27. Given a triangle, circumscribe an equilateral triangle of greatest area.

28. Construct a quadrilateral, given its sides and the line segment joining the mid-points of the diagonals.

29. Given a triangle ABC and a straight line g passing through C, find a point X on g, from which AC and BC are visible at equal angles.

30. Given a triangle ABC and a point D on the straight line AB, find a point X on the straight line AC, from which AD and DB are visible at the same angle.

31. Given a straight line g and two points A and B on opposite sides of it, find a point X on g, so that $AX + XB$ is the least.

32. Inscribe in a square an equilateral triangle, given one of its vertices.

33. Describe a circle touching a given one, and passing through two given points.

Chapter XVIII

MEASURING LENGTHS, AREAS AND VOLUMES

1. Measuring Line Segments

By the "measure axiom" for line segments, each segment is of certain positive length. If a point C on a straight line AB is between A and B, then the length of the line segment AB equals the sum of those of the line segments AC and BC. Thus, the axiom requires that each line segment should be associated with a certain value, the above additive property being valid. No measurements of the line segment are assumed. The question naturally arises as to the relation of the results of a practical measurement which we normally make to the segment length whose existence is stated by the axiom.

Recall how a measurement is performed practically. Let AB be a given line segment. We take a standard of length, e.g., of one meter, make one of its ends coincident by a motion with that of the line segment, say, A, and mark the point A_1 where the other end of the standard goes. Similarly, we mark points A_2, A_3, ... If one of the points A_n so marked coincides with B, then we will say that the length of the line segment is n metres. This is the practical result of the measurement. Does it coincide with the number associated with the line segment by the measure axiom?

To prove that it does, we see that the length of AA_2 is equal, by the measure axiom, to the sum of those of AA_1 and A_1A_2. Since a motion preserves the lengths of line segments, $A_1A_2 = AA_1 = 1$.

Therefore, the length of $AA_2 = 2$, which corresponds to the results of the practical measurement if $B \equiv A_2$. It is proved similarly that if $B \equiv A_n$, the practical measurement result coincides with the length of AB, prescribed by the measure axiom.

It may happen that B does not coincide with any of A_n. Then there are such neighbouring points A_{n-1} and A_n that B is in $A_{n-1}A_n$. In practical measurement, we say that the length of AB is between $n - 1$ and n metres. If we speak of the length determined by the measure axiom, then the result is the same. In fact, by the measure axiom, the length of AB equals the sum of those of AA_{n-1} and $A_{n-1}B$. Hence, it is greater than $n - 1$. Similarly, we conclude that it is less than n.

For more precise measurement of the length of a line segment in practice, we divide the standard into 10, or some other number, equal parts, and perform the measurement by one of the known methods. Analysis which we omit here shows that the result of the practical measurement coincides with the one following from the measure axiom.

In connection with the practical measurement of the length of a line segment by cutting off a standard of length, the natural question arises as to what entails the existence of such a point A_n that the point B belongs to the line segment AA_n? It is not hard to give an answer. AA_n is of length n. And when n is sufficiently large, the length of AB is less than n (meaning the length of the line segment, determined by the measure axiom). Hence, B belongs to AA_n. Thus, that B belongs to AA_n for sufficiently large n (and, therefore, the possibility to measure line segments in practice) follows from the properties of real numbers, viz., for any number $d > 0$, there exists a natural number n such that $d \leqslant n$.

We have drawn the reader's attention to this circumstance, because, with another axiomatic construction of geometry, e.g., due to H. Hilbert, where the concept of the length of a line segment is not basic, and obtained in the measurement process, the existence of the point A_n is introduced as an axiom (*Archimedes' axiom*).

Note another circumstance in connection with the practical measurement of a line segment. If the measurement process does not stop after a finite number of steps, then we obtain two sequences of points P_n and Q_n possessing the following properties, (i) the point B is between P_n and Q_n, (ii) the lengths of the line segments AP_n form a nondecreasing sequence, whereas those of the line segments AQ_n a nonincreasing one, (iii) the length of the line segment P_nQ_n is $1/10^n$. By the property of real numbers, both sequences have the same limit. Since the length of AB is greater than that of AP_n, and less than that of AQ_n, this common limit is the length of AB. Thus, the practical measurement method always yields the length of the line segment, prescribed by the measure axiom.

2. Length of a Circumference

At school, the discussion of the length of a circumference starts with visual imagery. The student is asked to imagine a thread in circular form, cut it, and pull at its ends. Then the length of the obtained line segment is that of the circumference. Further, it becomes clear from the visual imagery that the length of a circumference can be made as little different as we please from the perimeter of the inscribed convex polygon with sufficiently small sides. On the basis of this proposition, it can be then proved in a perfectly strict manner that the ratio of the length of a circumference to its diameter is independent of the circumference, i.e., is the same for any two circumferences.

The defect of this treatment is that we do not give a definition of the concept of the length of a circumference, and then introduce a proposition requiring proof, which is caused by purely methodical argument. The concept of the length of a circumference assumes familiarity with that of the limit or supremum of a sequence. They seem complicated to the intermediate school student, and the proofs cannot be grasped at all.

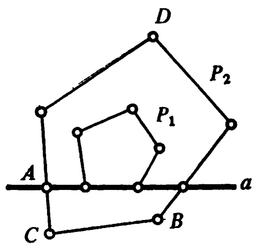


Fig. 144

The rigorous treatment of the problem of the length of a circumference is in the following. First, we define the concept. Viz., the *length of a circumference* is the supremum of the perimeters of convex polygons inscribed in a circumference, or the least number greater than the perimeter of any of them. To make the definition correct, or to define the length of a circumference, it is required that the perimeters of the inscribed polygons should be all bounded. The latter is proved by the following theorem.

If a convex polygon P_1 lies inside a convex polygon P_2, then the perimeter of P_1 is not greater than that of P_2. If P_1 is not coincident with P_2, then its perimeter is less than that of P_2.

Proof. Draw a straight line a containing one side of P_1 (Fig. 144). P_1 is on one side of this line, whereas P_2 either on the same side of a or there are points on P_2, lying on opposite sides, in which case a breaks P_2 into two polygons. Let Q_2 be the one in the same half-plane with P_1 relative to a. It contains P_1, and has perimeter less than that of P_2. In fact, the passage from P_2 to Q_2 is related to the replacement of the broken line by the line segment AB joining its ends.

Performing the same construction with each side of P_1, we finally obtain P_1. Hence, if P_1 does not coincide with P_2, then its perimeter is less than that of P_2.

Q.E.D.

We now prove another proposition from the school textbook. Viz., *the length of a circumference is as little different as we please from the perimeter of the inscribed convex polygon with sufficiently small sides.*

Proof. First, we see that, for any $\varepsilon > 0$, a convex polygon can be inscribed in a given circle so that its perimeter is different from the length of the circumference by not more than ε. In fact, assume that the statement is false. Then the perimeter of any inscribed polygon is not greater than $l - \varepsilon$, l being the length of the circumference. Therefore, the number l is not the least value greater than the perimeter of any inscribed polygon. $l - \frac{\varepsilon}{2}$ is less than l, and also greater than the perimeter of any inscribed polygon; a contradiction, and the statement is thus proved.

Now, let P be a polygon inscribed in the circle, with perimeter different from the length of the circumference by not more than $\varepsilon > 0$, and P' an inscribed polygon with sides less than δ. Complete P' with the vertices of P. The polygon P'' so formed has perimeter not less than P'. On the other hand, it is not greater than l. If we omit in P'' the segments of the broken line, meeting at the vertices of P, then its perimeter decreases, but not more than by $2n\delta$, where n is the number of vertices of P. Hence, the perimeter of P' is not less than $l - \varepsilon - 2n\delta$. Since after the choice of ε, n is fixed, $l - \varepsilon - 2n\delta$ can be made as little different from l as we please for sufficiently small ε and δ.

Q.E.D.

With the given definition of the length of a circumference, the question arises how it is related to that of the length of a curve, viz., as the limit of the lengths of broken lines inscribed in the curve, which we made use of in Ch. IX. It turns out that the above definition leads to the same result. In fact, we have proved that the perimeters of convex polygons inscribed in a circumference can be made as little different from its length as we please if their sides are sufficiently small. This means that the length of a circumference is the limit of the perimeters of inscribed convex polygons if the lengths of their sides decrease arbitrarily.

3. Areas of Figures

The school treatment of the topic of area starts with the discussion of crops on two plots, one in the form of a square, and the other of arbitrary form. This argument is followed by the conclusion regarding the existence of area and its properties, viz., additivity and equality for congruent figures. Further, proceeding from the existence of area and on the basis of its properties, we can rigorously deduce formulas for the areas of simple figures such as a rectangle, parallelogram or triangle.

The strict theory of area should be started with the proof of the following theorem.

On a set of simple figures admitting partition into a finite number of non-overlapping triangles, i.e., without common interior points, a function S called area can be defined so that it possesses the following properties, viz.,

(i) *for figures with interior points,* $S > 0$,

(ii) *if a figure G is made up of two figures* G_1 *and* G_2 *having no interior points in common, then* $S(G) = S(G_1) + S(G_2)$,

(iii) *congruent figures have equal areas,* and

(iv) *for a square with unit side,* $S = 1$.

The function S satisfying conditions (i)-(iv) *is unique.*

Proof. We define the area S as follows. Put $S = \frac{1}{2}ah$ for a triangle, where a is its side, and h the altitude on it. For any figure G, the quantity S is determined as the sum of the areas of triangles in any of its partition. To make the above definition correct, it is required that the area of a triangle should not depend on the side taken or the altitude drawn, and that the area of a figure, defined in terms of the addition of areas of the component triangles, should not depend on a partition into them.

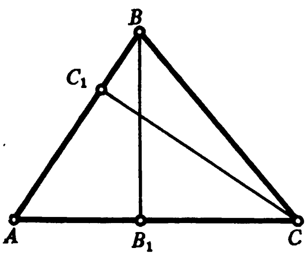


Fig. 145

First, we prove that the area of a triangle does not depend on the side taken and corresponding altitude. Let ABC be a given triangle (Fig. 145). Draw its altitudes CC_1 and BB_1. The right triangles AC_1C and AB_1B are similar, since the angle A is common. Hence,

$$\frac{AC}{AB} = \frac{CC_1}{BB_1}, \quad AC \cdot BB_1 = AB \cdot CC_1.$$

Therefore, we obtain the same result not depending on the side AC and altitude BB_1, or the side AB and altitude CC_1.

We now prove that, in partitioning a triangle into smaller ones, its area equals the sum of those of the component triangles irrespective of the partitioning method.

First, we consider the partition in Fig. 146, where the triangle ABC is broken into triangles CAD_1, CD_1D_2, CD_2D_3, . . ., all of them with the same altitude h from their common vertex C. It is also that of the triangle ABC.

The sum of the areas of the triangles is

$$\frac{AD_1 \cdot h}{2} + \frac{D_1D_2 \cdot h}{2} + \frac{D_2D_3 \cdot h}{2} + \ldots = \frac{(AD_1 + D_1D_2 + D_2D_3 + \ldots) \cdot h}{2}.$$

Since $AD_1 + D_1D_2 + D_2D_3 + \ldots = AB$, the sum of the areas is $\frac{AB \cdot h}{2}$, or the area of the triangle ABC.

We now consider an arbitrary partition of the triangle ABC into smaller ones. Assume that any two triangles in the partition either

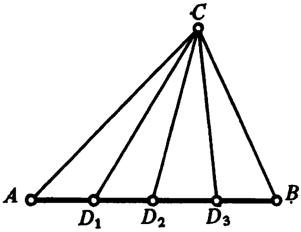


Fig. 146

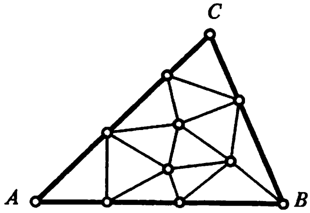


Fig. 147

have no common points, or have a common vertex, or a common side. E.g., such a partition is shown in Fig. 147.

In Fig. 148, another partition triangle PQR is shown. Its areá can be represented as the algebraic sum of those of the three triangles APQ, AQR, ARP obtained from the triangle PQR by replacing one of the vertices with A. The sign of the areas in the sum is determined by the following rule. If a vertex to replace A is on one side with it relative to the straight line joining the other two vertices, then the area of the triangle is taken with a plus; if it is on the other side, then with a minus. If, replacing with A, three points are in one straight line, then the addend is omitted, i.e., the area is assumed to be zero.

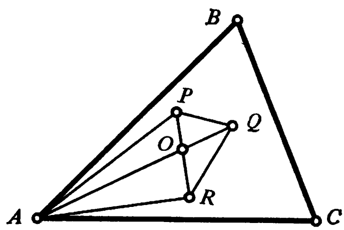


Fig. 148

E.g., consider the position of the triangle PQR in Fig. 148.

As we have proved,

$$S(PQR) = S(PQO) + S(QRO),$$
$$S(APQ) = S(APO) + S(PQO),$$
$$S(ARQ) = S(ARO) + S(QRO),$$
$$S(APR) = S(APO) + S(ARO).$$

Hence,

$$S(PQR) = S(APQ) + S(ARQ) - S(ARP).$$

The correctness of our statement regarding the representation of the area of the triangle PQR as the algebraic sum of those of the triangles APQ, AQR and ARP has been verified by a concrete example of the position of the triangle PQR. We could also consider other possibilities for its position, and see that our statement is always valid.

Representing the area of each partition triangle as the algebraic sum of those of triangles with A as a vertex, we add together the areas of all the triangles in the partition, and obtain the sum of those of triangles AXY, where XY is a side of a partition triangle. If XY is inside the triangle ABC, then the area of the triangle AXY is involved in the sum twice, because XY is the side of two triangles in the partition. Since they are on opposite sides of the straight line XY, the area of the triangle AXY is once with a plus, and once with a minus, thus eliminating each other.

If the line segment XY is on the side BC of the triangle ABC, then the area of the triangle AXY is only once involved in the sum, with a plus. However, if the side XY is on AB or AC, then the area of AXY simply is zero. Eventually, the sum of the areas of the triangles in our partition is that of the triangles AXY with sides XY on the side BC of the triangle ABC. It has been proved earlier that it is equal to the area of the latter, and thus equals the sum of the areas of the triangles in any partition.

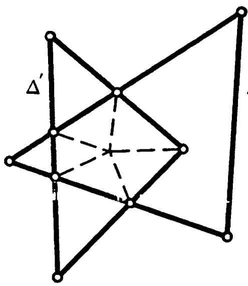


Fig. 149

Now, let a simple figure F be in one case partitioned into triangles Δ_1', Δ_2', Δ_3', . . . , and, in another case, into triangles Δ_1'', Δ_2'', Δ_3'', ... We prove that the sums of the areas of the triangles in the first and second partitions are equal.

The triangles in the first and second partition, taken together, divide F into convex polygons, viz., triangles, quadrilaterals, pentagons and hexagons, each of which is the part common to one triangle in the first partition and another in the second. (One such pentagon is shown in Fig. 149.) Break them into triangles Δ_1''', Δ_2''', Δ_3''', . . ., and do it so that every two either have no common points at all, or have a common vertex, or a common side.

It has been proved that each Δ_k' of the first partition of F equals the sum of the areas of the triangles Δ_k''' involved. Similarly, each triangle Δ_k'' of the second partition is represented as the sum of Δ_k'''. Therefore, the sums of the areas of the triangles both of the first and second partitions of F equal that of the areas of Δ_k'''. Hence, the sums of the areas of the first and second partition triangles are equal, i.e., the area of F is independent of the way it is partitioned into triangles.

We now prove that the area so defined, in fact, possesses properties (i)-(iv). The first of them is obvious. To prove the second, suppose that the figure G is partitioned into two figures G_1 and G_2 without common interior points. Let G_1 be partitioned into triangles Δ'_k, and G_2 into triangles Δ''_k. We then obtain a partition of G into Δ'_k and Δ''_k. The area of G_1 equals the sum of the areas of Δ'_k, whereas that of G_2 is the sum of the areas of Δ''_k. The area of G is the sum of the areas of Δ'_k and Δ''_k. Therefore, it equals that of the areas of G_1 and G_2, and the second property is thus proved.

The third property of area follows from the equality of the areas of congruent triangles (the corresponding sides are equal, and the altitudes on them are also equal).

We now prove the fourth property. A square with unit side is divided by a diagonal into two right triangles with unit sides containing the right angles, the area of each being $1 \cdot 1/2$. Therefore, the area of the square is 1.

Finally, we prove that area is determined uniquely by properties (i)-(iv), which has been actually proved in the school textbook, where it was shown on the basis of these properties that the area of a rectangle with sides a and b equals ab and that the area of a triangle is one-half the product of its base and altitude. The uniqueness in the definition of the area of a triangle implies that in the definition of the area of any simple figure, and the theorem is thus proved completely.

We now define the concept of area for any figure. We will say that a figure G possesses certain area if, for any $\varepsilon > 0$, there exists a simple figure G_1 containing G, and a simple G_2 contained in G, whose areas differ by not more than ε. For figures with area in the sense of the above definition, the value of the area $S(G)$ can be defined as the infimum of those of simple figures containing G or as the supremum of the areas of simple figures contained in G. The area so defined for figures having area possesses properties (i)-(iv). However, we do not give the proof here.

A simple sufficient test of the existence of area for a figure is that its boundary should have zero area; in particular, if the boundary of the figure consists of rectifiable curves.

In the school course of geometry, they usually consider figures which are bounded by straight line segments or circles. They all possess area in the sense of the above definition.

4. Volumes of Solids

The school treatment of the topic of the volume of solids also starts similarly, viz., with a clear proof of the existence of volume, and its properties of additivity and equality for congruent solids. A strict treatment of the topic assumes the proof of the following theorem.

On a set of simple solids admitting partition into a finite number of disjoint triangular pyramids, a function V called volume can be defined to possess the following properties, viz.,

(i) *for solids with interior points,* $V > 0$,

(ii) *if a solid T is made up of two solids T_1 and T_2 without interior points in common, then*

$$V(T) = V(T_1) + V(T_2),$$

(iii) *congruent solids have equal volumes,*

and

(iv) *for a cube with unit edge,* $V = 1$.

The function V satisfying conditions (i)-(iv) *is unique.*

In principle, proof is not different from that of the corresponding theorem for the areas of simple figures. Viz., the volume of a simple solid is defined as the sum of those of triangular pyramids composing it, whereas that of a pyramid is defined by the formula $V = \frac{1}{3} Sh$, where S is the area of its base, and h the altitude on it. The correctness of the definition is then proved, or the independence of the volume of a triangular pyramid from the choice of its base, as well as that of the volume of a simple solid from its partition into triangular pyramids.

The proof that the volume definition is correct is followed by the verification of (i)-(iv), and, finally, by that of the uniqueness of volume.

The concept of volume is defined for any solids as follows. We will say that a solid T has certain volume if, for any $\varepsilon > 0$, there exists a simple solid T_1 containing T and a solid T_2 contained in T, whose volumes differ by not more than ε. For a solid T with volume in the sense of this definition, its volume $V(T)$ is defined either as the infimum of the volumes of simple solids containing T or as the supremum of the volumes of simple solids contained in it. The volume so defined of solids (which have volume) satisfies conditions (i)-(iv).

A simple sufficient test for the existence of volume of a solid is that its boundary should have zero volume. In the school course of geometry, they consider solids bounded by pieces of planes, and of cylindric, conic or spherical surfaces. It is easy to see that each of them can be contained by a simple solid of arbitrarily small volume. Therefore, solids bounded by such surfaces do have certain volume. In school treatment, the existence of volume in the sense of the above definition is usually implied in deducing the formula for volume.

5. Area of a Surface

The school textbook supplies the following definition of the area of a surface. Let F be a surface, and F_δ the set of points in space, which are from the surface at a distance not greater than δ. We call the limit of the ratio $V(F_\delta)/2\delta$ as $\delta \to 0$ the *area of the surface F*. This definition is explicitly clear, especially after the example of the amount of paint necessary for a surface to be coated and a square lamina. It also possesses the advantage that it makes very simple the deduction of formulas for the areas of surfaces studied at school, viz., of a sphere, spherical cap, spherical zone, cylinder and cone.

However, the question arises as to the definition relation to that of the area of a surface, given in higher school, and, in particular, to the one from Ch. XI. We now show that both lead to the same formula for the area of a surface.

We now introduce curvilinear coordinates u, v, w in the neighbourhood of a surface F as follows. Cut off a line segment of length $|w|$ on the normal to F at a point (u, v), and take the values u, v, w as the coordinates of its end, w being positive on one side of the surface, and negative on the other. The Cartesian coordinates of the point (x, y, z) are certain functions of u, v, w. To make the passage from coordinates x, y, z to u, v, w possible, it is required that the Jacobian J should not vanish, viz.,

$$J = \begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix} \neq 0.$$

We show that this holds in a sufficiently small neighbourhood of the surface, or for sufficiently small w. We will assume the surface regular and, at least, twice differentiable.

If we denote the position vector of a point on the surface by $\mathbf{r}(u, v)$, and the unit normal vector by $\mathbf{n}(u, v)$, then

$$J = ((\mathbf{r} + w\mathbf{n})_u\ (\mathbf{r} + w\mathbf{n})_v\ (\mathbf{r} + w\mathbf{n})_w).$$

For $w = 0$, $J = (\mathbf{r}_u\mathbf{r}_v\mathbf{n}) = |\mathbf{r}_u \wedge \mathbf{r}_v| \neq 0$. Therefore, $J \neq 0$ also in a certain neighbourhood of the surface, i.e., for sufficiently small $|w|$.

Now, let the curve γ bounding the surface be rectifiable, and of length l. Divide it into l/δ equal parts (without loss of generality, we assume l/δ integral). Construct cubes with centres at each division point, and edges 4δ. Their total volume is not greater than $\frac{l}{\delta}(4\delta)^3$. Let F'_δ be that part of the solid F_δ, which is filled with the normals of length δ to the surface F. Its volume is different from that of F_δ by not more than $\frac{l}{\delta}(4\delta)^3$. Therefore, for certain θ, $0 \leqslant \theta \leqslant 1$, we have

$$V(F_\delta) = V(F'_\delta) + \theta \frac{l}{\delta}(4\delta)^3,$$

$$\frac{1}{2\delta} V(F_\delta) = \frac{1}{2\delta} V(F'_\delta) + \theta 32 l\delta.$$

Passing to the limit as $\delta \to 0$, we obtain

$$S(F) = \lim_{\delta \to 0} \frac{1}{2\delta} V(F_\delta) = \lim_{\delta \to 0} \frac{1}{2} V(F'_\delta).$$

We now calculate the limit on the right-hand side, viz.,

$$\lim_{\delta \to 0} \frac{1}{2\delta} V(F'_\delta) = \lim_{\delta \to 0} \frac{1}{2\delta} \iiint\limits_{F,\ |w| \leqslant \delta} J\, du\, dv\, dw$$

$$= \iint\limits_{F} \left[\lim_{\delta \to 0} \frac{1}{2\delta} \int\limits_{-\delta}^{\delta} (\mathbf{r}_u + w\mathbf{n}_u \mathbf{r}_v + w\mathbf{n}_v \mathbf{n})\, dw\right] du\, dv = \iint\limits_{F} (\mathbf{r}_u \mathbf{r}_v \mathbf{n})\, du\, dv.$$

Thus,

$$S(F) = \iint\limits_{F} (\mathbf{r}_u \mathbf{r}_v \mathbf{n})\, du\, dv = \iint\limits_{F} |\mathbf{r}_u \wedge \mathbf{r}_v|\, du\, dv,$$

and we obtain the same formula for the area of a surface, which was derived in Ch. XI with another definition of area.

Chapter XIX

ELEMENTS OF PROJECTION DRAWING

1. Representation of a Point on an Epure

A solid is represented on the plane by means of projection with parallel straight lines. Usually, its projection onto one plane does not lead to a full image. Therefore, two or even three projections onto two or three planes, respectively, are used. We consider the representation of a solid by means of orthogonal projection onto two planes.

Let H and V be two planes meeting at right angles in a straight line x (Fig. 150). For convenience, we will assume H horizontal, and V vertical. A solid is orthogonally projected onto H and V. The projection of a solid onto the horizontal plane is said to be *horizontal*, whereas that on the vertical *vertical*. H and V are called the *projection planes*, and x the *axis of projection*. On projecting the figure onto H and V, we turn H through 90° about x until it coincides with V. Both projections will then be in one plane. The drawing so obtained with the representation of both projections is called an *epure*.

Consider the position of the horizontal and vertical projections of an arbitrary point on the epure. The following property is valid.

The vertical and horizontal projections of a point on the epure are represented as points in the straight line perpendicular to the axis of projection.

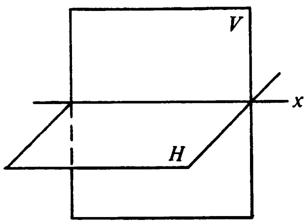


Fig. 150

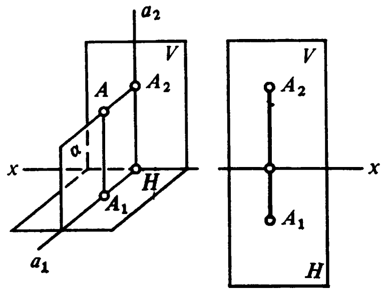


Fig. 151

Proof. Draw a plane α perpendicular to the axis of projection x through the given point A to cut the planes H and V in two straight lines a_1 and a_2 (Fig. 151). The horizontal projection A_1 of A is in a_1, since the perpendicular from A to H is in α. Similarly, the vertical projection A_2 of A is in a_2. The straight lines a_1 and a_2 are perpendicular to x. Like any motion, a rotation is angle-preserving, and a_1, a_2 are made coincident when H and V coincide after the rotation. Thus, the projections of A are represented as points of a_2 on the epure.

2. Problems Leading to a Straight Line

Given a straight line a by its projections on the epure, and the horizontal projection of a point A in a, find the vertical projection of A.

Solution. Let a_1 and a_2 be the horizontal and vertical projections of a, and A_1 the horizontal projection of A (Fig. 152). The vertical projection of A is in the straight line perpendicular to the axis of projection passing through A_1; it is also on the vertical projection a_2 of a. Therefore, it is the point where the straight lines meet.

Given a straight line a and an outside point A by its projections on the epure, construct the projections of the straight line passing through A, and parallel to a.

Solution. Since parallel straight lines have parallel projections, the projections of the required straight line are obtained if we draw

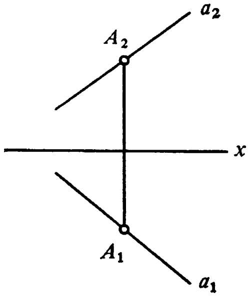


Fig. 152

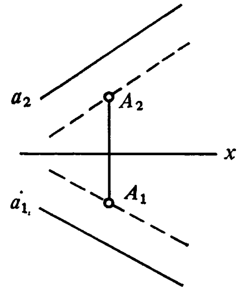


Fig. 153

through the projections of A straight lines parallel to the corresponding projections of a (Fig. 153).

3. Determination of the Length of a Line Segment

Find the length of a line segment AB, given its projections on the epure.

Solution. If AB is parallel to one of the projection planes, e.g., the vertical plane, then its length equals that of the projection onto it. We shall learn whether or not AB is parallel to the vertical plane from its horizontal projection which should be parallel to the axis of projection.

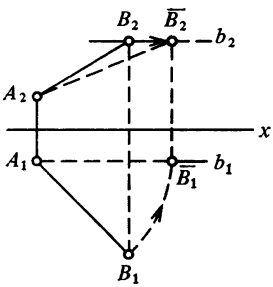


Fig. 154

Assume that AB is not parallel to any of the projection planes. Rotate AB about the straight line projecting A onto the horizontal plane. The projections of B will then vary. Viz., the horizontal projection of B moves along a circle with centre at a point A_1, and the vertical projection along a straight line b_2 parallel to the axis of projection passing through a point B_2 (Fig. 154).

When the line segment is parallel to the vertical plane, the projection of B_1 falls on a straight line parallel to the axis of projection, passing through A_1. Denote B_1 in this position by $\overline{B}_1$. The line segment $A_1\overline{B}_1$ is the horizontal projection of a line segment equal to AB, and parallel to the vertical plane. Its vertical projection $A_2\overline{B}_2$

is not hard to find. The vertical projection of the end B of the line segment rotated is in the intersection of the straight line passing through $\bar{B}_1$, and perpendicular to the axis of projection and b_2. As indicated above, AB is equal to $A_2\bar{B}_2$.

4. Problems Leading to a Straight Line and a Plane

Let H and V be two projection planes, and α an arbitrary plane intersecting the planes in two straight lines h and v, respectively (Fig. 155), called the *traces* of α on the projection planes. Viz., h is called the *horizontal trace*, and v the *vertical trace*.

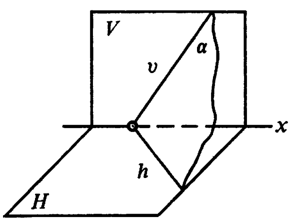


Fig. 155

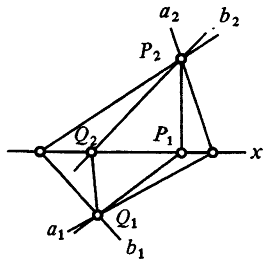


Fig. 156

The traces of a plane intersect on the axis of projection or are parallel to it if the plane is parallel to the axis. If the plane is parallel to one of the projection planes, then it possesses one trace. It is vertical if the plane is parallel to the horizontal, and horizontal if it is parallel to the vertical plane. Planes are represented as their traces on the epure.

Find the straight line where two planes meet, given by their traces on the epure, i.e., determine the projections of the straight line.

Solution. Let α and β be the given planes, a_1 and a_2 the traces of α, and b_1 and b_2 those of β (Fig. 156). The straight line c, in which α and β intersect, cuts the vertical plane at a certain point P. Its vertical projection P is the point where the plane vertical traces meet, i.e., a_2 and b_2, while the horizontal projection P_1 is on the axis of projection.

Similarly, c cuts the horizontal plane at a point Q whose horizontal projection Q_1 is the point where a_1 and b_1 meet, while the vertical projection is on the axis of projection. The required projections of c are obtained if we join the point Q_2 to P_2 (vertical projection) and P_1 to Q_1 (horizontal projection).

Given a straight line by its projections on the epure, find the traces of the plane passing through the line perpendicular to the given projection plane, e.g., H.

Solution. Since the plane is perpendicular to H, its horizontal trace coincides with the horizontal projection of the given line, whereas the vertical trace is perpendicular to the axis of projection. To obtain the vertical trace, a straight line should be drawn perpendicular to the axis of projection through the point where the line horizontal projection meets the axis (Fig. 157).

Given the projections of a straight line and the traces of a plane, find the point where the line meets the plane, i.e., the projections of the point.

Solution. Draw through the given straight line a plane perpendicular to H, and find a straight line h in which the plane intersects

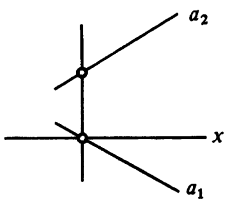


Fig. 157

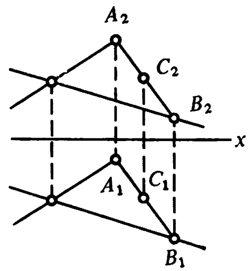


Fig. 158

the given one. Similarly, we find the straight line v in which the given plane intersects the one passing through the given straight line perpendicular to the vertical plane. The projections of the required point are the points where the corresponding projections of h and v meet.

Given the projections of two intersecting straight lines and the horizontal projection of a point, find the vertical projection of the point if it is known to be in the plane determined by the given straight lines.

Solution. Draw an arbitrary straight line through the horizontal projection C_1 of the given point, intersecting the horizontal projections a_1 and b_1 of the given lines (Fig. 158). Denote the intersection points by A_1 and B_1, and draw through them straight lines perpendicular to the axis of projection. Denote by A_2 and B_2 respectively, the points where they meet the vertical projections of the given straight lines. The line segments A_1B_1 and A_2B_2 are the horizontal and vertical projections of the line segment with ends on the given straight lines. Hence, the vertical projection C_2 of the required point is where the straight line passing through C_1 perpendicular to the axis of projection meets A_2B_2.

5. Representation of a Prism and a Pyramid

Solving solid geometry problems, we often have to represent solids by their parallel projections onto a plane. The general theory used in this case is known from the school course of geometry.

Viz.,

(i) Straight line segments are represented on the projection plane as straight line segments.

(ii) Parallel line segments of a figure are represented as parallel line segments.

(iii) The ratio of line segments on one straight line or parallel straight lines is preserved in parallel projection. In particular, the

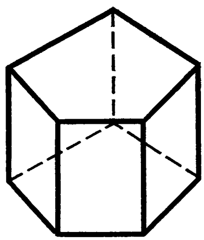


Fig. 159

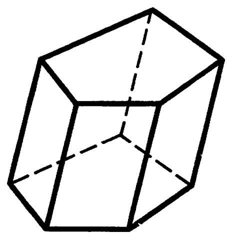


Fig. 160

mid-point of a line segment is represented as that of its projection.

These rules are necessary, because their violation is always conspicuous.

We now consider the representation by parallel projection of the most frequently represented solids, prisms and pyramids. The lateral edges of a prism are parallel and equal; therefore, they are represented as parallel line segments equal in length. In the case of a right prism, its lateral edges are usually represented as vertical line segments. Since the lateral faces of a prism are parallelograms, and parallelism is preserved in parallel projection, they are represented as parallelograms in the projection plane. Thus, to represent a right prism with a given polygon as the base, we have to draw parallel straight lines through its vertices, cut off equal line segments on them, and join their ends in the same sequence as on the base (Fig. 159).

To represent an oblique prism, we do the same, with the only difference that the lateral edges are drawn to be parallel to each other, but not vertical (Fig. 160). Anyway, we have to see to it that the edge projections should not overlap. Otherwise, the representation is not convincing. For better impression, the edge projections not visible by the observer can be represented in dotted lines.

The base of a triangular prism is represented on the projection plane as an arbitrary triangle, whereas that of a parallelepiped must, naturally, be a parallelogram. In representing the base of a prism, we should generally resort to the above rules. In particular, the parallel sides of a base should be represented as parallel line segments, while the projection of a point-symmetric base should also be point-symmetric.

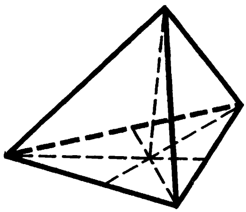


Fig. 161

In representing the base of a pyramid, we should follow the same rules as for the base of a prism. The height of a regular pyramid is represented as a vertical line segment, whereas its foot as the base circumcentre. If the pyramid is triangular, then the circumcentre of its base is at the point of intersection of the three medians (Fig. 161).

6. Representation of a Cylinder, a Cone and a Sphere

In representing a cylinder and a cone, it is most difficult to draw their basis. As the projections of circles, they are represented as ellipses. To construct an ellipse with a given major axis, we can first construct a circle on the major axis as on diameter (Fig. 162), decrease proportionally the vertical half-chords, e.g., twice, and join the obtained points with smooth curves. If we take sufficiently many points, then the ellipse representation is quite accurate. Normally, in solving problems, we confine ourselves to four points, the ends of the semi-axes. Drawing an ellipse through them is simplified by knowing the directions of tangents.

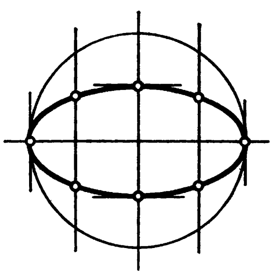


Fig. 162

To construct a cylinder with the obtained base, we draw several generators through the base points, cut off equal intercepts on them, and join their ends with a smooth curve. The extreme generators on the cylinder projection touch the bases.

To inscribe a regular polygon in the base of a cylinder, it is first inscribed in the circle from which the ellipse is obtained, and then vertical straight lines are drawn through the vertices to meet the ellipse (Fig. 163).

The obtained points are the vertices of the required polygon. We then easily construct the prism inscribed in the cylinder with this base.

In constructing a prism circumscribed about a cylinder, we should remember that the sides of the prism bases are tangent to the cylinder bases, and the corresponding points of tangency on the upper and lower bases are the ends of a generator.

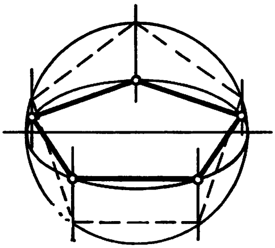


Fig. 163

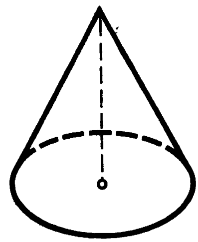


Fig. 164

To represent a cone, we first construct its base as of a cylinder, viz., draw the cone height from the centre of the ellipse as a vertical line segment, and then, from the cone vertex, extreme generators so that they touch the base (Fig. 164).

In the case of a sphere, parallel projection is assumed orthogonal to the projection plane. The sphere, therefore, is represented as a circle.

7. Construction of Sections

Solving solid geometry problems, it is often necessary to construct a section of a solid on its representation. Here, we give certain hints which can be used in such a construction. First of all, note that the section of a prism by a plane parallel to its lateral edges is a parallelogram whose sides in the lateral faces are parallel to lateral edges. The section of a cylinder by a plane parallel to its axis is a rectangle represented as a parallelogram whose opposite sides are two generators of the cylinder. A section of a pyramid (or cone) with a plane passing through the vertex is a triangle whose one vertex is that of the pyramid (resp. cone), and the other two are on the base contour.

The section of a prism or a cylinder by a plane parallel to the bases is congruent to the base, and obtained from it by a translation. The section of a pyramid or a cone with a plane parallel to the base is homothetic to the base with respect to the vertex. This permits us to construct sections with such planes easily.

To construct a solid with a plane in general position is more difficult. Consider the principal case, given a straight line g in which the secant plane meets that of the prism base. E.g., a section passes through a side of the base (Fig. 165). Let a point A be given on the prism edge, and a secant plane pass through it.

We draw the plane of the face with A. It intersects the base in a straight line. Let B be the point where the straight line meets g. The straight line AB is in the secant plane and the plane of the face.

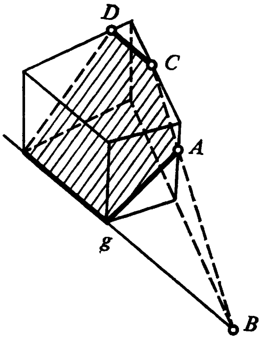


Fig. 165

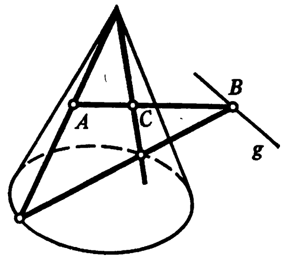


Fig. 166

Therefore, the line segment AC of this line, which is in the prism face, is a side of the required section.

We then find a point D in the next face, upper base. The line segment CD should be parallel to g. Proceeding further, we find all the vertices of the polygonal section, and thus construct the section itself. For convenience, the section is sometimes shaded.

The section of a pyramid with a plane in general position is constructed similarly. First, the intersection of the secant plane with the base is found, and then the procedure is the same as for a prism.

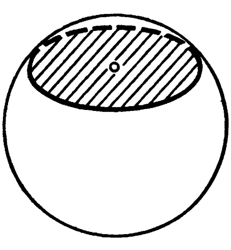


Fig. 167

Consider the section of a cone with a plane in general position, intersecting the plane of the base in a given straight line g. Suppose a point A is given on the lateral surface, and a secant plane passes through it (Fig. 166). Draw some plane through the vertex and A. It intersects the lateral surface in two generators. Let B be the intersection with g of this plane trace on the base plane. The intersection of AB with the generator is then a point C of the section. Any number of points in the section can be constructed thus. Joining them with a smooth curve, we obtain the section by the given plane.

The section of a cylinder by a plane is constructed similarly.

The section of a sphere by a plane is a circle, and its parallel projection an ellipse (Fig. 167).

EXERCISES TO CHAPTER XIX

1. Account for the following method of constructing the parallel projection of a regular hexagon. Viz., take the projections of every other vertex arbitrarily; find the point O where the medians of the triangle meet these projections, and then find the projections of the remaining three vertices symmetrically to the constructed with respect to O.

2. Given the parallel projection of a circle (ellipse) and the projection of one of its diameters, how can the projection of the perpendicular diameter be constructed?

3. Given the parallel projection of a circle, construct the projection of an inscribed square if one of its vertices is known.

4. Given the projection of a circle, how can that of a circumscribed square be constructed?

5. Given the parallel projection of a circle, construct the projection of the inscribed equilateral triangle if the projection of one of its vertices is known.

6. Given the projection of a circle, how can that of a circumscribed equilateral triangle be constructed?

7. Given the projection of a prism, construct its section passing through the lateral edge and a point in one of the faces if the projection of the point is known.

8. Given the parallel projection of a prism, construct its section passing through a base side and a point in one of the faces if the projection of the point is known.

9. Given the parallel projection of a prism, construct a section passing through two points on the sides of one of the bases and through a given point on one of the lateral edges.

10. Given the parallel projection of a prism, construct a section parallel to the bases, and passing through a given point in a lateral face.

11. Given the parallel projection of a regular triangular pyramid, construct the section passing through a lateral edge and the height.

12. Given the parallel projection of a triangular pyramid, construct the section passing through a base side, and dividing the height in a given ratio.

13. Given the parallel projection of a pyramid, how can the section passing through the vertex and two points on the base be constructed if their projections are known?

14. Given the parallel projection of a pyramid, how can the section parallel to the base, and passing through a point given in a lateral face, be constructed if the projection of the point is known?

15. Given the parallel projection of a pyramid, how can the section passing through three points on lateral edges be constructed?

16. Given the parallel projection of a cylinder, how can the projection of an inscribed (resp. circumscribed) regular quadrangular prism be constructed?

17. Given the parallel projection of a cylinder, how can the projection of an inscribed triangular (resp. hexagonal) prism be constructed? The same question for a circumscribed prism.

18. Given the parallel projection of a cone, how can the projection of an inscribed triangular (resp. hexagonal) pyramid be constructed? The same question for a circumscribed pyramid.

19. Given the parallel projection of a cone, how can an inscribed (resp. circumscribed) regular quadrangular pyramid be constructed?

20. Given the parallel projection of a cylinder (resp. cone), how can the section parallel to the base, and passing through a given point of the height, be constructed?

Chapter XX

POLYHEDRAL ANGLES AND POLYHEDRA

1. Cosine Law for a Trihedral Angle

Theorem. *Let* α, β *and* γ *be the face angles of a trihedral angle, and* C *the dihedral angle opposite to* γ. *Then*

$$\cos\gamma = \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos C.$$

Proof. Let S be the vertex of the trihedral angle, a, b, c its edges, α, β, γ the face angles made by the edges b and c, c and a, a and b, respectively, and C the dihedral angle at the edge c, i.e., opposite to γ (Fig. 168).

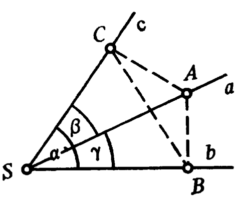


Fig. 168

First, we assume that α and β are acute. Cut off on c segment SC of unit length, and draw perpendiculars from C until they meet a and b at points A and B, respectively. Apply the cosine law to the triangles ABC and ABS.

We have

$$AC^2 + BC^2 - 2AC\cdot BC\cdot\cos C = AB^2,$$
$$SA^2 + SB^2 - 2SA\cdot SB\cdot\cos\gamma = AB^2,$$

or

$$\tan^2\alpha + \tan^2\beta - 2\tan\alpha\tan\beta\cos C$$

$$= \frac{1}{\cos^2 \alpha} + \frac{1}{\cos^2 \beta} - 2\,\frac{1}{\cos \alpha} \cdot \frac{1}{\cos \beta} \cdot \cos \gamma. \qquad (*)$$

Noticing that

$$\frac{1}{\cos^2 \alpha} - \tan^2 \alpha = 1, \quad \frac{1}{\cos^2 \beta} - \tan^2 \beta = 1,$$

we obtain from (*) that

$$\cos \gamma = \cos \alpha \cos \beta + \sin \alpha \sin \beta \cos C.$$

If α is obtuse, and β acute, then we have to take the intersection of the perpendicular to c with a produced. The relation (*) from which to express $\cos \gamma$ still holds, since α is replaced by $180° - \alpha$, C by $180° - C$, and γ by $180° - \gamma$. Similarly, (*) is valid also if β is obtuse.

Q.E.D.

2. Trihedral Angle Conjugate to a Given One

Let a, b and c be the edges of a trihedral angle with vertex S. The plane of the angle (bc) separates the space into two half-spaces with the half-line a in one of them. Draw the half-line a' from S perpendicular to the plane of the angle (bc), directed into the half-space complementary to that with a. Similarly, construct the half-lines b' and c' perpendicular to the planes of the angles (ac) and (ab), respectively. The trihedral angle whose edges are the half-lines a', b' and c' is said to be *conjugate* to the original angle (abc) (Fig. 169). It is easy to see that *the faces of a conjugate angle are perpendicular to the edges of the given one*. The conjugacy property is commutative, i. e., if a trihedral angle $(a'b'c')$ is conjugate to a trihedral angle (abc), then (abc) is conjugate to $(a'b'c')$. We conclude from the property of angles whose sides are perpendicular each to each that the face angles of a conjugate angle and the corresponding dihedral angles of the given trihedral angle are supplementary. Viz., the face angle $(b'c')$ and the dihedral angle at the edge a are supplementary, etc. Similarly, dihedral angles of a conjugate trihedral angle and the corresponding face angles of the given one are supplementary. In particular, the dihedral angle with the edge a' and the plane angle (bc) are supplementary.

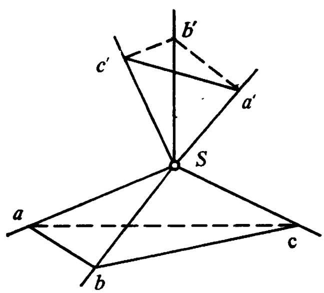


Fig. 169.

Theorem. *Let A, B, C be the dihedral angles of a trihedral angle, and γ the face angle opposite to C.*

Then

$$\cos C = -\cos A \cos B + \sin A \sin B \cos \gamma.$$

It is a simple corollary to the cosine law for a trihedral angle conjugate to a given one.

3. Sine Law for a Trihedral Angle

Theorem. *Let* α, β *and* γ *be the face angles of a trihedral angle, and* A, B, C *the opposite dihedral angles. Then*

$$\frac{\sin \alpha}{\sin A} = \frac{\sin \beta}{\sin B} = \frac{\sin \gamma}{\sin C}.$$

Proof. Cut off on the edge c a unit line segment SC (Fig. 170). Drop from C the perpendicular on the plane of the angle (ab). Denote its foot by $\bar{C}$, and draw from C planes perpendicular to the edges a and b. Denote by A and B the points where they meet a and b or their extensions.

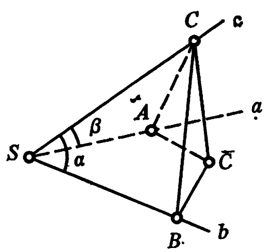


Fig. 170

We now find the length of the perpendicular $C\bar{C}$. From the right triangle SCB with the right angle at B, we obtain

$$CB = 1 \cdot \sin \alpha.$$

Now, from the right triangle $CB\bar{C}$ with the right angle at $\bar{C}$, we find the length of $C\bar{C}$. Viz.,

$$C\bar{C} = CB \sin B = \sin \alpha \sin B.$$

The length of $C\bar{C}$ can be found differently, from the right triangle ACS and $CA\bar{C}$, viz.,

$$C\bar{C} = \sin \beta \sin A.$$

Comparing the expressions for $C\bar{C}$, we find

$$\sin \alpha \sin B = \sin \beta \sin A.$$

Hence,

$$\frac{\sin \alpha}{\sin A} = \frac{\sin \beta}{\sin B}.$$

Similarly, we obtain the relation

$$\frac{\sin \beta}{\sin B} = \frac{\sin \gamma}{\sin C}.$$

Q.E.D.

4. Relation Between the Face Angles of a Polyhedral Angle

Theorem. *Any two face angles of a convex trihedral angle are together greater than the third.*

Proof. Let α, β and γ be the face angles of a trihedral angle. Show that $\gamma < \alpha + \beta$. If $\alpha + \beta \geqslant 180°$, then the statement is obvious, since $\gamma < 180°$. Let $\alpha + \beta \leqslant 180°$. Applying the Cosine Law to the trihedral angle, we get

$$\cos \gamma = \cos \alpha \cos \beta + \sin \alpha \sin \beta \cos C.$$

Since $\cos C > -1$, and $\sin \alpha$ and $\sin \beta$ are positive,

$$\cos \gamma > \cos \alpha \cos \beta - \sin \alpha \sin \beta,$$

its right-hand side being nothing but $\cos(\alpha + \beta)$. Thus, $\cos \gamma >$

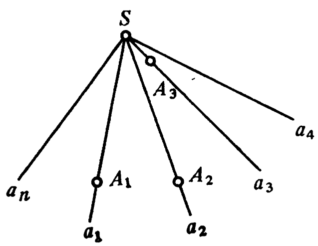


Fig. 171

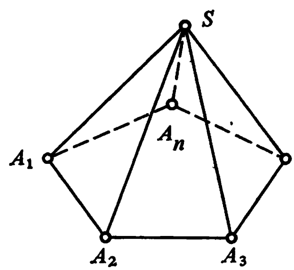


Fig. 172.

$\cos(\alpha + \beta)$. We know that the cosine decreases as the angle increases from 0° to 180°. Hence, $\gamma < \alpha + \beta$.

Q.E.D.

Theorem. *The sum of the face angles of a convex polyhedral angle is less than* 360°.

Proof. Let $a_1, a_2, \ldots, a_n$ be the edges of a convex polyhedral angle with vertex S. Mark two points A_1 and A_2 on the sides a_1 and a_2. Now, take a point A_3 on the side a_3, sufficiently near to S, and draw a plane α through A_1, A_2 and A_3 (Fig. 171). If A_3 is sufficiently near to S, α intersects all $a_1, a_2, a_3, \ldots, a_n$. Let $A_1, A_2, \ldots, A_n$ be the points where α meets the edges of the angle, vertex S. It follows from the convexity of the polyhedral angle that the polygon P with vertices $A_1, A_2, A_3, \ldots, A_n$ is convex, too (Fig. 172).

Consider the polyhedral angle, vertex S, and trihedral angles with vertices $A_1, A_2, \ldots, A_n$. The sum of all its face angles consists of that of the angles of P, i.e., $180°n - 360°$, and the angle-sums of the triangles A_1A_2S, A_2A_3S, $\ldots$, A_nA_1S, or $180°n$. Thus, the sum of all face angles is $2 \cdot 180°n - 360°$.

The angle of P is less than the sum of the other two angles for each trihedral angle, vertex A_k. Therefore, the above sum is greater than $(180°n - 360°)\,2 + \theta$, where θ is the sum of the face angles at S, i.e.,

$$(180°n - 360°)\,2 + \theta < 2\cdot 180°n - 360°.$$

Hence, $\theta < 360°$.

Q.E.D.

5. Area of a Spherical Polygon

Let V be a convex polyhedral angle. Take the unit sphere with centre at the vertex. The figure P obtained when the sphere intersects V is called a convex *spherical polygon*. The points where the edges of the angle meet the sphere are called *vertices*, and the arcs of the great circles obtained by the intersection of the faces with the sphere are called *sides*. The angles of the polygon α_k equal the dihedral angles of V.

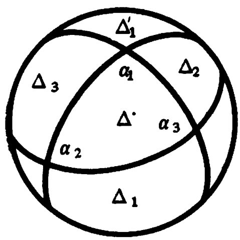


Fig. 173

The area of a spherical polygon P can be found by the Gauss-Bonnet theorem. We have

$$\sum_k (\pi - \alpha_k) = 2\pi - \iint_P K\,dS.$$

Since $K = 1$,

$$S(P) = \sum \alpha_k - \pi(n-2),$$

where n is the number of sides. In particular,

$$S = \alpha_1 + \alpha_2 + \alpha_3 - \pi$$

for a spherical triangle.

We now give the elementary deduction of the formula for the area of a spherical polygon. We start with a triangle. Let V be a trihedral angle whose faces break the sphere into eight triangles symmetric about the centre of the spheres (Fig. 173). Let Δ be the spherical triangle in the intersection of V with the sphere, and α_1, α_2, α_3 its angles. The figure formed by the triangles Δ and Δ_1 is part of the sphere contained inside a dihedral angle equal to α_1.

Therefore, its area is

$$S(\Delta) + S(\Delta_1) = \left(\frac{\alpha_1}{2\pi}\right) 4\pi = 2\alpha_1.$$

Similarly, we obtain

$$S(\Delta) + S(\Delta_2) = 2\alpha_2,$$
$$S(\Delta) + S(\Delta_3) = 2\alpha_3.$$

The sum of the areas of the triangles Δ_1', Δ_2, Δ_3 and Δ equals the area of the hemisphere, 2π, whereas Δ_1' is symmetric to Δ_1, and is, therefore, equal to it in area.

We obtain

$$S(\Delta) + S(\Delta_1) + S(\Delta_2) + S(\Delta_3) = 2\pi.$$

Adding the first three equalities together termwise, and subtracting the fourth one, we get

$$2S(\Delta) = 2\alpha_1 + 2\alpha_2 + 2\alpha_3 - 2\pi.$$

Hence,

$$S(\Delta) = \alpha_1 + \alpha_2 + \alpha_3 - \pi.$$

Q.E.D.

Now, let V be a polyhedral angle. Draw planes through one edge and each of the others, intersecting the sphere in great circles which break P into triangles similarly to a plane polygon triangulated by diagonals emanating from one vertex. If we write the obtained formulas for the area of each triangle, and add them termwise, then we get the area of the polygon $S(P)$ on one side, and the sum of its angles and $-\pi(n-2)$ on the other.

Thus,

$$S(P) = \sum \alpha_k - \pi(n-2).$$

Q.E.D.

6. Convex Polyhedra. Concept of Convex Body

According to the definition given at school, a *polyhedron* is a solid bounded by a finite number of planes. This should be understood in the sense that the whole boundary of the polyhedron, or its *surface*, is in these planes. A polyhedron is said to be *convex* if it is on one side of each of the bounding planes, i.e., in one of the half-spaces determined by the plane. The following theorem gives a clear idea of the structure of a convex polyhedron.

A convex polyhedron is made by the intersection of a finite number of half-spaces with a common interior point. Conversely, the intersection of a finite number of half-spaces, if bounded and having an interior point, is a convex polyhedron.

Proof. Let P be a solid bounded by a finite number of planes α_k, i.e., a convex polyhedron, and A its interior point. Each α_k separates space into two half-spaces. Suppose that E_k is the half-space with A, E_k being closed, i.e., $\alpha_k \subset E_k$.

We state that the intersection P' of E_k is P. In fact, let $X \in P$. It also belongs to each E_k, and to their intersection; therefore, $P \subset P'$.

Now, let $X \in P'$. Show that $X \in P$. Since A is an interior point of P, points of the line segment AX, which are near to A, also belong to P. If X is not in P, then AX intersects its surface at a certain point Y belonging to one of α_k. Therefore, A and X are on opposite sides of α_k, which is contrary to the definition of P', and the first statement of the theorem is thus proved.

Proof of the second statement is quite simple. The intersection of a finite number of half-spaces, if bounded and having interior points, is a solid bounded by a finite number of planes; therefore, it is a convex polyhedron, and the theorem is thus proved completely.

A bounded closed set with interior points, which contains, along with any two of its points, the line segment joining them, is called a *convex* body. It is obvious that a convex polyhedron is a convex body. An example of a convex body which is not a convex polyhedron may be given by a sphere. In general, any solid bounded by a closed regular surface with non-negative Gaussian curvature is convex. It can be proved that any convex body is representable as the intersection of a number of half-spaces. Generally speaking, the set of these half-spaces is infinite.

Similarly, the concept of plane *convex domain* is defined as the set of points, which contains, along with any two of its points, the line segment joining them. Any plane convex domain is the intersection of a number of half-planes. For a convex polygon, their number is finite.

7. Euler Theorem for Convex Polyhedra

The Euler theorem in question has been proved in Sec. 6, Ch. XII, by the Gauss-Bonnet theorem. Viz., we have proved that, *for any convex polyhedron,*

$$\alpha_0 - \alpha_1 + \alpha_2 = 2,$$

where α_0 is the number of vertices, α_1 the number of edges, and α_2 the number of faces.

We now give a simple elementary proof.

Let P be a convex polyhedron, and F its face. Take an interior point in F, and shift it a little outside. Project the polyhedron onto the plane of the face from this point. F then transforms into itself, and the remaining part of P is projected inside. The projections F_k of the faces break F into convex polygons (Fig. 174).

The angle-sum of a polygon F_k is

$$\sigma_k = \pi n_k - 2\pi, \qquad (*)$$

where n_k is the number of sides. To find the sum of the angles of all F_k, including F, we add all (*) together termwise. The second addend on the right-hand side is then repeated α_2 times, i.e., the number of

F_k, including F. The sum of the first addends is $2\pi\alpha_1$. The factor 2 appears, because each side belongs to two polygons.

Thus,

$$\sigma = 2\pi\alpha_1 - 2\pi\alpha_2.$$

We now find the angle-sum σ in another way, first adding the angles of the polygons at a common vertex. Meanwhile, if a vertex is inside F, then the sum of the angles at this vertex is 2π. However, if it is one of those of F, then the sum of the angles at this vertex is twice the corresponding angle of F. Therefore, σ can be represented in the form

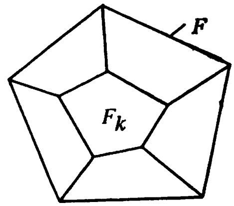


Fig. 174

$$\sigma = 2\pi\alpha_0 - \Sigma\, 2\,(\pi - \beta_k),$$

where β_k are the angles of F, and summation is over the vertices of F. The value $\pi - \beta_k$ is an exterior angle of F. Since the sum of the exterior angles of a convex polygon is 2π,

$$\sigma = 2\pi\alpha_0 - 4\pi.$$

Comparing the expressions obtained for σ, we obtain the Euler formula

$$\alpha_2 - \alpha_1 + \alpha_0 = 2.$$

Q.E.D.

8. Cauchy Theorem

Convex polyhedra equidecomposable into congruent faces are congruent. (By a convex polyhedron, we understand the surface, and not the solid.)

Proof. Assume the contrary, that there exist two non-congruent convex polyhedra P_1 and P_2 equidecomposable into congruent faces. It is then obvious that P_1 will have edges with dihedral angles different from the corresponding angles of P_2. We assign a plus or a minus to each of the edges according as the dihedral angle is greater or less than the corresponding one of P_2. It is evident that if a distinguished edge emanates from a certain vertex, then at least another distinguished edge necessarily emanates from the same vertex, too. Therefore, the distinguished edges have no free vertices, and break P_1 into domains g. If all g are homeomorphic to the circle, then the Euler characteristic is

$$\chi = \alpha_0 - \alpha_1 + \alpha_2 = 2,$$

where α_0 is the number of the distinguished vertices (from which the distinguished edges emanate), α_1 that of the distinguished edges, and α_2 that of g.

If there are domains g non-homeomorphic to the circle, then $\alpha_0 - \alpha_1 + \alpha_2 > 2$, since our partition can be completed with new sides without altering the number of domains and vertices (Fig. 175). Their introduction only decreases $\alpha_0 - \alpha_1 + \alpha_2$, and when all the domains are homeomorphic to the circle, we have $\alpha_0 - \alpha_1 + \alpha_2 = \chi = 2$. Thus, for our partition of the polyhedron into domains,

$$\alpha_0 - \alpha_1 + \alpha_2 \geqslant 2.$$

The boundary of each domain g is a broken line whose segments are assigned either a plus or a minus. We assume an angle at the vertex of g distinguished if its sides have opposite signs. We now estimate the total number of the distinguished angles of the domains g. The number of the distinguished angles is not greater than n if n is even

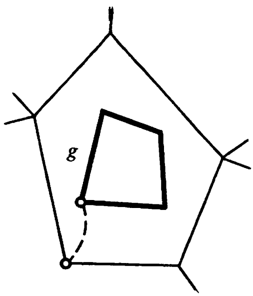


Fig. 175

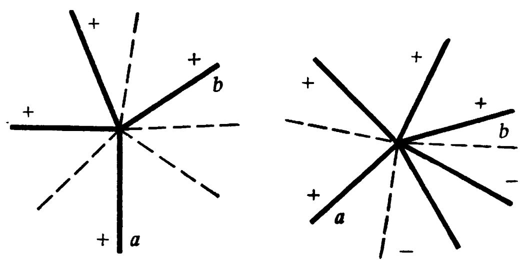


Fig. 176

for a domain with n sides, and not greater than $n - 1$ if n is odd. Therefore, the total number of the distinguished angles is

$$\omega \leqslant 2a_3 + 4a_4 + 4a_5 + 6a_6 + \ldots,$$

where $a_3, a_4, a_5, \ldots$ are the numbers of domains with three, four, five sides, etc.

Since each distinguished edge belongs to two domains,

$$\sum na_n = 2\alpha_1,$$

$$4\alpha_1 - 4\alpha_2 = \sum (2n-4)\, a_n = 2a_3 + 4a_4 + 6a_5 + 8a_6 + \ldots$$

Hence,

$$\omega \leqslant 4\alpha_1 - 4\alpha_2.$$

To obtain a lower estimate to ω, we show that the number of the distinguished angles at a given vertex is not less than 4. If it is less than 4, then the distinguished edges are either prescribed the same sign, or the edges prescribed opposite signs do not alternate, i.e., in walking around the vertex of an angle, we first meet the edges with one sign, and then with the other (Fig. 176, where the non-distinguished edges are shown in dotted lines). To prove that both of these possibilities are improbable, we suppose that a and b are the extreme edges with a plus. It is clear intuitively that, in passing from a given polyhedral angle of the polyhedron P_1 to the corresponding angle of the polyhedron P_2, the angle (ab) must decrease if we consider that part of polyhedral angle, to which the "positive" edges belong, and, vice versa, must increase if we consider the part with the "negative" edges. However, if there are no distinguished edges in the part under consideration, then the angle (ab) remains unaltered. Anyhow, if there are distinguished edges in the polyhedral angle, we then come to a contradiction. It remains to supply a strict proof.

Thus, let a convex polyhedral angle $V = (a_1a_2 \ldots a_n)$ be transformed into a convex polyhedral angle V' with an increase of dihedral angles at the edges $a_2, a_3, \ldots, a_{n-1}$, and preserving the num-

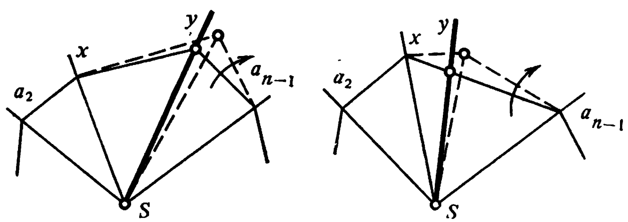


Fig. 177.

ber of the face angles $(a_1a_2), (a_2a_3), \ldots, (a_{n-1}a_n)$. We show by induction that the face angle (a_1a_n) then increases, the statement for trihedral angles following from the Cosine Law.

Consider the convex angles $V_{xy} = (xa_2 \ldots a_{n-1}y)$ obtained from V if the number of dihedral angles at the edges a_2 and a_{n-1} grows, but not greater than in passing from V to V'. We choose V_{xy} such that the face angle (xy) is greatest. Show that the dihedral angle at least at one of its edges a_2 or a_{n-1} is the same as in V'. In fact, if both angles are less, then V_{xy} can be slightly deformed, increasing the face angle (xy) (Fig. 177), which is contrary to the choice of V_{xy}.

In passing from V_{xy} to V', the plane angle (xy) increases by the induction hypothesis, since the two corresponding dihedral angles are congruent, which permits us to reduce the problem to the case of $(n - 1)$-hedral angles. Finally, first passing from V to V_{xy}, and then from V_{xy} to V', we conclude that the face angle (a_1a_n) increases in passing from V to V', which is just what was required to show.

We now complete the proof of the Cauchy theorem. Since the number of the distinguished angles is not less than 4 at each distinguished vertex,

$$\omega \geqslant 4\alpha_0.$$

Comparing the inequality with the above $\omega \leqslant 4\alpha_1 - 4\alpha_2$, we have $4\alpha_0 \leqslant 4\alpha_1 - 4\alpha_2$, i.e., $\alpha_0 - \alpha_1 + \alpha_2 \leqslant 0$, which is contrary to the above relation $\alpha_0 - \alpha_1 + \alpha_2 \geqslant 2$.

Q.E.D.

A convex polyhedron can be cut into a finite number of convex polygons. The question naturally arises, given a finite number of convex polygons, can a convex polyhedron be glued together from them only by deformation? It turns out that it is always possible if the sides to be glued together are of the same length, and the angle-sum of the polygons whose vertices coincide on gluing is not greater than 2π (*A. D. Alexandrov theorem*, in whose proof the Cauchy theorem is used essentially).

9. Regular Polyhedra

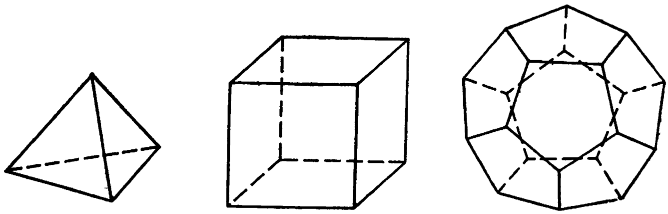
According to the school definition, a convex polyhedron is said to be *regular* if its faces are regular polygons with the same number of sides, and the same number of edges meet at each vertex.

The faces of a regular polyhedron may be either equilateral triangles, or squares, or regular pentagons. Indeed, starting with a regular hexagon, the interior angles are not less than 120°, and, since not less than three edges meet at each vertex, the sum of the face angles would then be not greater than $3 \cdot 120° = 360°$ at the vertex of a regular polyhedron, which is impossible, because we know that the sum of the face angles of any convex polyhedral angle is less than 360°.

If the faces of a regular polyhedron are equilateral triangles, then the number of edges at a vertex must not be greater than 5. In fact, if it is greater than five, the sum of the face angles at the vertex of the polyhedron is not less than 360°, which is impossible. Thus, the number of edges meeting at a vertex of a regular polyhedron with triangular faces must only be three, or four, or five. That in a regular polyhedron with square or pentagonal faces can only be three.

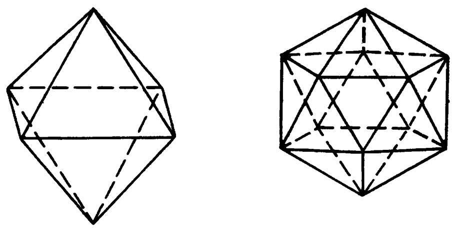
To find all regular convex polyhedra, we start with those with three edges meeting at each vertex. It follows from the Cosine Law for a trihedral angle that the dihedral angles are congruent in such a polyhedron, and uniquely expressed in terms of face angles. Therefore, proceeding from some vertex, and consecutively completing the faces, we come to three regular polyhedra, viz., a *tetrahedron*, *cube* and *dodecahedron* (Fig. 178).

If more than three edges meet at a vertex of a regular polyhedron, in which case the faces are triangles, then the problem gets more complicated. Nevertheless, it is not hard to construct two of such polyhedra. In one, called an *octahedron*, the vertices are the centres of the faces of a cube, and, in the other called an *icosahedron*, the vertices are the centres of the faces of a dodecahedron. Four edges meet at each vertex of an octahedron, and five of an icosahedron (Fig. 179).



Tetrahedron Cube Dodecahedron

Fig. 178



Octahedron Icosahedron

Fig. 179

The question arises, can there be any other regular polyhedra with triangular faces, in which four edges meet at each vertex, similarly to an octahedron, or five as in an icosahedron. It turns out that there are no other regular polyhedra of this form, which follows from the Cauchy theorem stating that convex polyhedra equidecomposable into congruent faces are congruent.

EXERCISES TO CHAPTER XX

1. Find the dihedral angles at the base of a regular n-sided pyramid if the lateral face angles at the base are α.
2. Find the angles of the lateral faces of a regular n-sided pyramid if the base dihedral angles are β.
3. Find the dihedral angles at the base of a regular n-sided pyramid if the lateral faces make an angle γ with the base.
4. Find the dihedral angles at the lateral faces of a regular n-sided pyramid if the base dihedral angles are β.

5. Find the dihedral angles at the base of a regular n-sided pyramid if the dihedral angles at the lateral edges are δ.

6. Find the dihedral angles at the lateral edges of a regular n-sided pyramid if the vertex face angles are φ.

7. Find the angle between the edge of trihedral angle and the plane of the opposite face, given the face (resp. dihedral) angles.

8. Given the dihedral angles at one vertex of an oblique parallelepiped, how can the dihedral angle at any other vertex be found?

9. Given the base angles and those formed by a lateral edge with the base sides at the common vertex of an oblique triangular prism, how can the angles formed by the other lateral edges with the base sides be found?

10. Find the dihedral angles of the regular convex polyhedra: a tetrahedron, an octahedron, a dodecahedron, and an icosahedron.

11. Find the circumradii and inradii of the convex regular polyhedra: a tetrahedron, a cube, an octahedron, a dodecahedron and an icosahedron.

12. How many different methods are there to make a regular polyhedron (tetrahedron, cube, octahedron, dodecahedron and icosahedron) coincident with itself?

ANSWERS TO EXERCISES, HINTS AND SOLUTIONS

Chapter I

1. 2. 2. (2, 0). 3. (0, 3). 4. A straight line parallel to the y-axis and separated from it by 3 units. 5. The ends of AB lie in different half-planes relative to the y-axis, but in one half-plane relative to x. 6. Positive. 7. 4 (3). 8. 2. 9. —2. 10. A straight line that contains the bisectors of the first and third quadrants. 11. A straight line which contains the bisectors of the second and fourth quadrants. 12. (a) On straight lines parallel to the y-axis separated from it by a; (b) on the bisectors of the coordinate angles. 13. (a) In a strip bounded by straight lines parallel to the y-axis and separated from it by a; (b) within a rectangle with centre at the origin of coordinates and sides $2a$ and $2b$ parallel to the coordinate axes. 14. $(x, -y)$; $(-x, y)$; $(-x, -y)$. 15. The coordinates of the point symmetric to A (x, y) about the bisector of the first (second) quadrant will be y and x (or, respectively, $-y$, $-x$). 16. If we interchange the coordinate axes, then A (x, y) will have the abscissa y, and the ordinate x. 17. $AB = 5$, $AC = 10$, $BC = 5$. 18. Compare the distances between the points. Point B lies between A and C. 19. (4, 0). 20. (3, 3) and (15, 15). 21. The third vertex C of the triangle lies at the distance AB from A and B: $C\left(\frac{2+\sqrt{3}}{2}, \frac{1+2\sqrt{3}}{2}\right)$, or $C\left(\frac{2-\sqrt{3}}{2}, \frac{1-2\sqrt{3}}{2}\right)$. 22. Make use of the fact that in a square the sides are equal and the diagonals are $\sqrt{2}$ times greater than the sides. Answer: (a) C $(1, \sqrt{2})$, D $(\sqrt{2}, 1)$; (b) C $(-1, 0)$, D $(0, -1)$. 23. Make use of the Pythagoras theorem. If A (x_1, y_1), B (x_2, y_2), C (x_3, y_3) are the vertices of the triangle with a right angle (C), then $(x_3-x_1)^2+(y_3-y_1)^2+(x_3-x_2)^2+(y_3-y_2)^2=(x_2-x_1)^2+(y_2-y_1)^2$. 24. If A (x_1, y_1), B (x_2, y_2), C (x_3, y_3) are the vertices of the triangle, then $(x_3-x_2)^2+(y_3-y_2)^2>(x_3-x_1)^2+(y_3-y_1)^2$. This follows from the fact that in a triangle a longer side is opposite a larger angle. 25. Find the centre O of the circle circumscribed about the triangle ABC, and compare the radius of this circle with the distance from the centre to the point D. 26. The coordinate notation of the "inequality of the triangle". The inequality means that the distance between (a, b) and (a_1, b_1) is not larger than the sum of their distances to (a_2, b_2). 27. Make use of the fact that the diagonals of a parallelogram bisect each other. Answer: D (2, —1), O (2, 1). 28. (0, —2). See the previous problem. 29. (3, 3). 30. Show that the quadrilateral is a parallelogram (Ex. 27). Compare the lengths of the sides and diagonals. 31. Make use of the fact that the medians are divided at the intersection point in the ratio 2 : 1 counting from the vertices. Answer: $(x_1+x_2+x_3)/3$. 32. The mid-points of the sides of the triangle and one of its vertices are the vertices of a parallelogram. Answer: A $(x_1-x_2+x_3, y_1-y_2+y_3)$, B $(x_3-x_1+x_2, y_3-y_1+y_2)$, C $(x_2-x_3+x_1, y_2-y_3+y_1)$. 33. The vertices of the original triangle divide in the ratio λ: $(1-\lambda)$ the segments connecting (x_0, y_0) with the vertices of the given triangle. Answer: $((1-\lambda)x_0+\lambda x_1, (1-\lambda)y_0+\lambda y_1)$, $((1-\lambda)x_0+\lambda x_2, (1-\lambda)y_0+\lambda y_2)$, $((1-\lambda)x_0+\lambda x_3, (1-\lambda)y_0+\lambda y_3)$. 34. Make use of the geometric considerations associated with the division of a line segment in a given ratio. 35. Let

(x_1, y_1) and (x_2, y_2) be the ends of one line segment, and (x_3, y_3) and (x_4, y_4) are those of the other. If the segments intersect, the point of intersection divides the first segment in the ratio $\lambda: (1-\lambda)$, and the second in the ratio $\mu: (1-\mu)$. The result is two representations for the coordinates of the intersection point: $(1-\lambda)x_1 + \lambda x_2 = (1-\mu)x_3 + \mu x_4$, and $(1-\lambda)y_1 + \lambda y_2 = (1-\mu)y_3 + \mu y_4$. The segments intersect if the solutions to this system in terms of λ and μ meet the conditions $0 < \lambda, \mu < 1$. **36.** Use the method of mathematical induction. **37.** $x_1 = 4$, $x_2 = -2$. **38.** (a) At $a = 0$, the centre of the circle lies on the axis of ordinates, (b) at $b = 0$, the centre lies on the axis of abscissas, (c) at $c = 0$, the circle passes through the origin of coordinates, (d) at $a = 0$, $b = 0$, the centre of the circle is at the origin of coordinates, (e) at $a = 0$, $c = 0$, the circle touches the axis of abscissas at the origin of coordinates. **39.** Notice that $(x-a)^2 + (x-b)^2$ is the square of the distance from (x, y) to the centre of the circle and apply the Pythagoras theorem to a right-angled triangle whose one side is a line segment of a tangent, and the other is the radius of the circle. **40.** Use the fact that for external points the degree is the square of the tangent, and for the internal points it is the square (with a minus sign) of the half-chord passing through the given point perpendicular to the diameter connecting this point with the centre of the circle. **41.** Let (x, y) be a point of the locus. Its distances from F_1 and F_2 are $\sqrt{(x-c)^2+y^2}$ and $\sqrt{(x+c)^2+y^2}$, respectively. The locus is described by the equation $\sqrt{(x-c)^2+y^2} + \sqrt{(x+c)^2+y^2} = 2a$. In order to reduce this equation to the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we will transpose the first radical to the right-hand side of the equation and square both sides. We get $(x+c)^2+y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2 + y^2$. We leave the radical on the right-hand side of the equation and transpose the other terms to the left-hand side. Then after some simplifications we obtain $cx - a^2 = -a\sqrt{(x-c)^2+y^2}$. Squaring both sides we obtain, after simple transformations, $a^4 - a^2c^2 = a^2y^2 + (a^2-c^2)x^2$, whence $\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1$, $a^2-c^2=b^2$. **42.** The problem is solved similarly to the previous one. The original equation is $\sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} = \pm 2a$. **43.** The equation of the locus is $\sqrt{(y-p)^2+x^2} = y$. Squared and simplified, the equation becomes $-2py + p^2 + x^2 = 0$. **44.** The equation of a curve in implicit form is $(x-a)^2 + (y-b)^2 = R^2$. It is seen from this that a and b are the coordinates of the centre and R is the radius. **45.** The equations of the curve are $x = \frac{a\lambda}{\lambda+\mu}\cos t$, $y = \frac{\mu a}{\lambda+\mu}\sin t$. At $\lambda = \mu$, the curve is a circle. **46.** The equation of the curve is $x = a\cos t + h\sin t$, $y = b\sin t + h\cos t$, where a, b, h and the parameter t have values as shown in Fig. 13. To derive these equations, represent the abscissa x and the ordinate y of the point on the curve as the algebraic sum of lengths of projections of the segments of the broken line $OABC$. **47.** The equations of the curve are $x = R\left(\frac{s}{R} - \sin\frac{s}{R}\right)$ and $y = R\left(1 - \cos\frac{s}{R}\right)$ (cycloid). The problem is solved as the previous one. Here the broken line is $OTSA$. **48.** Solving the equations $ax^2 + bxy + cy^2 + dx + ey = 0$ and $t = \frac{y}{x}$ for x and y, we obtain the parametric equations of the curve. **49.** $(x-1)^2 + (y-2)^2 = 4$. **50.** $(x+3)^2 + (y-4)^2 = 25$. **51.** The simultaneous equations $x^2 + y^2 + 2ax + 1 = 0$, $x = 0$ have no solutions. **52.** The given circle and the y-axis are tangent, since the simultaneous equations $x^2 + y^2 + 2ax = 0$, $x = 0$ have only one solution: $x = 0$, $y = 0$. **53.** The points of intersection of the circle with the x-axis are obtained by solving the simultaneous equations $x^2 + y^2 + 2ax + 2by + c = 0$, $y = 0$. The circle does not intersect the x-axis if the roots of the equation $x^2 + 2ax + c = 0$ are imaginary. The circle intersects the x-axis at two points, if the roots of this equation are real and different. The circle touches the axis if the roots coincide. **54.** The circles intersect at two points if $R_1 + R_2 > d$, where R_1 and R_2 are radii of the circles, and d is the distance between their centres. R_1, R_2 and d can be expressed in terms of the coefficients of the equations of the circles. We can also find these conditions, solving the system composed of the equations of these circles. **55.** The points of intersection of the circles are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. **56.** The point of intersection of the curves is (1, 0). **57.** If (x, y) obeys the equations of the curve, then the points $(-x, y)$ and $(x, -y)$, symmetric about the axes of coordinates, also obey these equations. Therefore, the points of intersection are symmetric about the axes of coordinates.

Chapter II

1. (1, —1), (2, —1), (1, 1). **2.** $a = b = 2$. **3.** Does not exist. **4.** Under the translation sending A into C, B is sent into B' of CD, with $BB' \parallel AC$. Therefore, B and B' are in one half-plane relative to AC, and B' is in the half-line CD, while the ray CB' coincides with $\overrightarrow{CD}$. **5.** See the hint to Ex. 4. **6.** $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{BC}$ are co-directional vectors, whereas $\overrightarrow{BA}$ is opposite to each of them. **7.** Apply the triangle inequality to A, B and C. **8.** See Ex. 7. **9.** The vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the corresponding coordinates equal. **10.** ± 12. **11.** 25. **12.** Under the rotation of all the vectors through $2\pi/n$, the sum is turned through the same angle. But, the vector system is transformed into itself. Therefore, their sum is zero. **13.** First, use the formula for the coordinates of the point A_0 where the medians meet, and prove that $\overrightarrow{A_0A} + \overrightarrow{A_0B} + \overrightarrow{A_0C} = 0$. Then use the representation for the vectors $\overrightarrow{OA} = \overrightarrow{OA_0} + \overrightarrow{A_0A}$, $\overrightarrow{OB} = \overrightarrow{OB_0} + \overrightarrow{B_0B}$, $\overrightarrow{OC} = \overrightarrow{OA_0} + \overrightarrow{A_0C}$. **14.** If the vectors have O (0, 0) as the origin, then their sum is zero. Then use the representation of the vector $\mathbf{r}_{mn} = \mathbf{r}^0_{mn} + \mathbf{r}_0$, where $\mathbf{r}_0$ is the vector from (x_0, y_0) to O, whereas $\mathbf{r}_{mn}$ from O to $(m\delta, n\delta)$. Answer: $\sum\limits_{m,n} \mathbf{r}_{mn} = -(2M+1) \times (2N+1)\,\mathbf{r}$. **15.** See the hint to Ex. 14. **16.** $\mathbf{b} = 0.5\mathbf{a}$. Therefore, the vectors $\mathbf{a}$ and $\mathbf{b}$ are co-directional. $\mathbf{d} = -0.5\mathbf{c}$. Hence, the vectors $\mathbf{c}$ and $\mathbf{d}$ have opposite directions. **17.** $\mathbf{b}$ (6, 8). **18.** $\mathbf{b}$ (—6, —8). **19.** 10. **20.** $\lambda_1 = \frac{1}{2}$, $\lambda_2 = -\frac{1}{2}$. **21.** The collinear vectors are $\mathbf{a}$ and $\mathbf{c}$, $\mathbf{b}$ and $\mathbf{d}$. **22.** The co-directional vectors are $\mathbf{a}$ and $\mathbf{c}$, and those with opposite directions $\mathbf{b}$ and $\mathbf{d}$; $|\mathbf{b}| = |\mathbf{c}|$, $|\mathbf{a}| = |\mathbf{d}|$. **23.** $n = 2$. **24.** $|\mathbf{a}| = |\mathbf{c}| = |\mathbf{d}| = 1$, the vectors $\mathbf{a}$ and $\mathbf{d}$ are collinear. **25.** $\mathbf{e}$ (0.6, 0.8). **26.** Compare the corresponding coordinates of the vectors $\overrightarrow{MN}$ and $\frac{1}{2}(\overrightarrow{AC} + \overrightarrow{BD})$. They are equal. **27.** (2, —3). **28.** $\lambda = -5$, $\mu = 4$. **29.** $\mathbf{ab} = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \cos\theta$, $\cos\theta \leqslant 1$. **30.** 90°. **31.** $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b})^2$. Answer: $\sqrt{3}$. **32.** 30°. **33.** $\cos A = 0.6$, $\cos B = 0$, $\cos C = 0.8$. **34.** $\angle A = 30°$, $\angle B = 60°$, $\angle C = 90°$. **35.** If $m = n = 0$, then the vectors are zero. If $m^2 + n^2 \neq 0$, then the vectors are perpendicular, for $\mathbf{ab} = 0$. **36.** $m = -\frac{8}{3}$. **37.** $\lambda = -1$. **38.** $\lambda = -\frac{1}{2}$. **39.** Find the scalar product of the vectors. **40.** See the hint to Ex. 39. **41.** Since $(\mathbf{a} + \mathbf{b}) \times$

$\times$ ($\mathbf{a} - \mathbf{b}$) = $\mathbf{a}^2 - \mathbf{b}^2 = |\mathbf{a}|^2 - |\mathbf{b}|^2 = 0$, we have $|\mathbf{a}| = |\mathbf{b}|$. **42.** First, show that the quadrilateral is a parallelogram, and then compare its diagonals. **43.** Prove that the quadrilateral is a parallelogram, and then compare its side with the diagonal. **44.** Make use of the fact that $\lambda^2\mathbf{a}^2 + 2\lambda\mu\mathbf{ab} + \mu^2\mathbf{b}^2 = (\lambda\mathbf{a} + \mu\mathbf{b})^2$.

Chapter III

1. $x + y - 2 = 0$. **2.** $\left(0, -\frac{3}{2}\right)$, $(-3, 0)$. **3.** $(1, -2)$. **4.** $x + 2y - 1 = 0$. **5.** $a = b = \frac{1}{3}$. **6.** $c = -3$. **7.** Use the fact that a straight line touches a circle if and only if it has with it only one point in common. Answer: $c = \pm\sqrt{2}$. **8.** The point of intersection of the first two straight lines obeys the third equation. **9.** The equations of the straight lines are not consistent. Multiplying the first one by 2, we obtain $2x + 4y = 6$, and from the second one $2x + 4y = 3$. There are no x and y satisfying both equations. **10.** $y = 3$. **11.** $3x - 2y = 0$. **12.** Use the fact that the straight line bisects the segment with the ends (x_1, y_1), (x_2, y_2). Answer: $x(y_1 + y_2 - 2y_0) - y(x_1 + x_2 - 2x_0) = x_0(y_1 + y_2) - y_0(x_1 + x_2)$.

13. The equation $\begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$ is linear in x and y and hence this is the equation of a straight line. The coordinates of all the three points (x_1, y_1), (x_2, y_2), (x_3, y_3) obey this equation. **14.** The equation admits of an equivalent form $(ax + by - c)(ax + by + c) = 0$. It is seen that this equation is satisfied by three points of the straight lines $ax + by + c = 0$, $ax + by - c = 0$, and only they. **15.** Let $A(b, d)$ be any point on the straight line and $\mathbf{e}(a, c)$ be a vector on the line; then for any point $P(x, y)$ on the line $\overrightarrow{OP} = \overrightarrow{OA} + t\mathbf{e}$. Hence, $x = b + at$, $y = d + ct$, $-\infty < t < \infty$. **16.** We will define the straight line by a parametric equation: $x = at + b$, $y = ct + d$. The equation $\omega(at + b, ct + d) = 0$ is satisfied for more than n various values of t. And since its degree is n, then it is an identity, i.e. it is satisfied for all t, and the line lies on the curve γ. **17.** If the equations of the circles are $x^2 + y^2 + 2a_1x + 2b_1y + c_1 = 0$ and $x^2 + y^2 + 2a_2x + 2b_2y + c_2 = 0$, then the equation of the locus of points of equal degrees will be $(x^2 + y^2 + 2a_1x + 2b_1y + c_1) - (x^2 + y^2 + 2a_2x + 2b_2y + c_2) = 0$. This equation is linear, therefore it is the equation of a straight line. The points of intersection of the circles obey it, since both parentheses vanish. **18.** $-\frac{c}{a} > 0 \left(\frac{c}{a} > 0\right)$. **19.** $\frac{c}{a} > 0$ and $\frac{c}{b} > 0$. **20.** If the point (x, y) obeys the first equation, then the point symmetric to it about the x-axis, i.e. point $(x, -y)$, obeys the second equation. Therefore, the straight lines are symmetric about the x-axis. **21.** If the point (x, y) obeys the first equation, then the point symmetric about the origin of coordinates, i.e. $(-x, -y)$, obeys the second equation. Therefore, the straight lines are symmetric about the origin of coordinates. **22.** The straight line in the pencil is parallel to the x-axis, if $a + \lambda a_1 = 0$ (the y-axis, if $b + \lambda b_1 = 0$). The straight line of the pencil passes through the origin of coordinates, if $c + \lambda c_1 = 0$. **23.** The sides of this triangle are the segments cut off by the straight line from the coordinate axes. The line produces an isosceles triangle if $|a| = |b|$. **25.** $y = \pm\sqrt{a^2 - b^2} - b$, $x = \pm\sqrt{a^2 - b^2} - a$. **26.** The vectors $\overrightarrow{(a, b)}$ and $\overrightarrow{(b, -a)}$ are perpendicular to the straight lines and to each other, since their scalar product is zero. **27.** $0°$. **28.** $\pm\sqrt{3}x + y =$

$= \frac{\sqrt{3}}{2} y = 0$. 29. $\cos^{-1} \frac{3}{\sqrt{10}}, \cos^{-1} \frac{3}{\sqrt{10}}, \pi - 2 \cos^{-1} \frac{3}{\sqrt{10}}$ 30. $\frac{a}{b} = -\frac{a_1}{b_1}$. 31. The vector $\overrightarrow{(a, c)}$ is parallel to the straight line. 32. $\cos\theta = \frac{|a_1c_1 + a_2c_2|}{\sqrt{a_1^2 + a_2^2} \cdot \sqrt{c_1^2 + c_2^2}}$. 33. The vertices of the quadrilateral are the points $\left(\pm \frac{c}{a}, \pm \frac{c}{b}\right)$. 34. The straight lines are given by the equations $ax \pm ay = b$, $cx \pm cy = d$. These straight lines are either parallel or perpendicular. 35. The vectors $\overrightarrow{(\alpha_1, \beta_1)}$ and $\overrightarrow{(\alpha_2, \beta_2)}$ are parallel to the straight lines. The parallelism condition for the lines is $\frac{\alpha_1}{\alpha_2} = \frac{\beta_1}{\beta_2}$. The perpendicularity condition for the lines is $\alpha_1\alpha_2 + \beta_1\beta_2 = 0$. 36. The vector $\overrightarrow{(a, b)}$ is perpendicular to the first straight line, and vector $\overrightarrow{(\alpha, \gamma)}$ to the second. Therefore, the parallelism condition is $a\alpha + b\gamma = 0$; the perpendicularity condition is $\frac{\alpha}{a} = \frac{\gamma}{b}$. 37. Make use of the parallelism and perpendicularity conditions discussed in Sec. 3. 38. Make use of the fact that substituting the coordinates of two points into the left-hand side of the equation gives an expression of the same sign, if the points lie on one side of the straight line, and of different signs, if the points lie on either side of the line. 39. Reduce the equation of one straight line to normal form and substitute into it the coordinates of any point of the other straight line. 40. See Ex. 39. 41. Make use of the equation for a pencil of lines. 42. Form the equation of the perpendicular bisector to the line segment with the ends (x_1, y_1), (x_2, y_2) and compare it with the equation $ax + by + c = 0$. 43. $x'y' = \frac{a^2}{2}$.

Chapter IV

2. Let A (ρ_1, θ_1) and B (ρ_2, θ_2) be the given points. From the Cosine Law in the triangle OAB: $AB^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\theta_2 - \theta_1)$. 3. ρ_0 is the distance from the pole to the straight line, α is the angle formed by the straight line $\rho \cos(\alpha - \theta) = \rho_0$ with the polar axis. 4. $\rho = R(1 - \cos\theta)$, where R is the radius of the circle. 5. $\rho = a\sqrt{2\cos 2\theta}$. 6. Repeat the reasoning for a plane intersecting a cone. The ellipse eccentricity is $\sin\alpha$. 7. The equation can be written in equivalent form $\rho = \frac{c}{1 + \sqrt{a^2 + b^2}\cos(Q - \alpha)}$, $\alpha = \tan^{-1}\frac{b}{a}$. Rotation of the polar axis by the angle α gives $\rho = \frac{c}{1 + \sqrt{a^2 + b^2}\cos\theta}$. The curve will be an ellipse, if $\sqrt{a^2 + b^2} < 1$, a hyperbola, if $\sqrt{a^2 + b^2} > 1$, a parabola, if $\sqrt{a^2 + b^2} = 1$. 8. From the conditions of the problem, find the constants a, b, c in the equation $\rho = \frac{c}{1 + a\cos\theta + b\sin\theta}$ (see Ex. 7). 9. Make use of the

equation for the conic section in polar coordinates. **10.** Inversion in polar coordinates with respect to the pole has the form $\rho' = \frac{1}{\rho}$, $\theta' = \theta$. **11.** Finding the points of intersection of the straight line with the conic section requires solving a quadratic equation. But it has not more than two roots. **12.** If the focus of a conic lies at the origin of coordinates, the equation of the conic becomes $x^2 + y^2 = \lambda (ax + by + c)^2$ ($ax + by + c = 0$ is the equation of the directrix). It follows that $\sqrt{x^2 + y^2} = \sqrt{\lambda}\,(ax + by + c) = \alpha x + \beta y + \gamma$. **13.** See Sec. 7, Ch. IV. **14.** See Sec. 7, Ch. IV. **15.** Pay attention to the fact that for the locus under consideration either the sum or the difference of the distances from the centres of the circles is constant. **16.** Form the equation of the locus of points of intersection. **17.** Form the equation of a curve obtained by the given construction. **18.** The normal form of the equation of the asymptotes is $\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\left(\frac{x}{a}+\frac{y}{b}\right) = 0$, $\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\left(\frac{x}{a}-\frac{y}{b}\right) = 0$. For the points on the hyperbola with the abscissa x $y = \pm b\sqrt{\frac{x^2}{b^2}-1}$. The distance from this point to the asymptotes will be $\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\left(\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right)$ and $\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\left(\frac{x}{a}-\sqrt{\frac{x^2}{a^2}-1}\right)$, or

$$\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}\left(\frac{x}{a}-\sqrt{\frac{x^2}{a^2}-1}\right)}, \quad \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}\left(\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right)}.$$

We see that the second expression decreases indefinitely as $|x| \to \infty$. **19.** The equation of the hyperbola can be written as $\left[\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\left(\frac{x}{a}+\frac{y}{b}\right)\right] \times \left[\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\left(\frac{x}{a}-\frac{y}{b}\right)\right] = \left(\frac{1}{a^2}+\frac{1}{b^2}\right)^{-1}$. The factors in square brackets are the distances from the points (x, y) to the asymptotes. We see that their product is constant and equal to $\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^{-1}$. 20. Form the equation of the projection of the circle, assuming that the plane passing through the centre of the circle is the plane of the projection, and the intersection of the xy-plane with the plane in which the circle lies is the x-axis. The equation of the projection of the circle will be $\frac{x^2}{R^2}+\frac{y^2}{R^2\cos^2\theta}=1$, where θ is the angle between the circle's plane and the xy-plane. **21.** Take the parabola arranged canonically relative to the coordinate system. **22.** The curves parallel to the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are given by the equations $\frac{x}{a}\pm\frac{y}{b}=c$. **23.** Make use of the result of Ex. 19. **24.** $\left(\frac{b^2}{\sqrt{a^2k^2+b^2}}, -\frac{a^2k}{\sqrt{a^2k^2+b^2}}\right)$ and $\left(-\frac{b^2}{\sqrt{a^2k^2+b^2}}, \frac{a^2k}{\sqrt{a^2k^2+b^2}}\right)$. **25.** The abscissas of the points of intersection of the tangents with the asymptotes $\left(\frac{a}{\frac{x_0}{a}-\frac{y_0}{b}}, \frac{b}{\frac{x_0}{a}-\frac{y_0}{b}}\right)$, $\left(\frac{a}{\frac{x_0}{a}+\frac{y_0}{b}}, -\frac{b}{\frac{x_0}{a}+\frac{y_0}{b}}\right)$ (where x_0, y_0 are the coordinates of the points of tangency) obey the conditions $\frac{1}{2}\left(\frac{a}{\frac{x_0}{a}-\frac{y_0}{b}}+\frac{a}{\frac{x_0}{a}+\frac{y_0}{b}}\right)=x_0$. For the ordinates holds a similar relation. Hence follows the statement of the problem. **26.** If x_1, x_2 are the abscissas of the points of intersection of the tangent with the asymptotes, and α is the angle formed by the asymptotes with the x-axis, then $S=\frac{1}{2}\left(\frac{x_1}{\cos\alpha}\right)\cdot\left(\frac{x_2}{\cos\alpha}\right)\cdot\sin 2\alpha=\frac{1}{2}\,\frac{a^2\sin 2\alpha}{\cos^2\alpha}$. **27.** The equation of the desired locus is $x^2+y^2=b^2+a^2$. **28.** See Exercise 27. **29.** Find the coordinates of the foci constructed and see that $c=\sqrt{a^2-b^2}$. **30.** See Sec. 7, Ch. IV. **31.** $\left(\frac{p}{2}, 0\right)$. **32.** The directrices of the ellipse and hyperbola are $x=\pm\frac{a}{e}$, where a is the major (real) semi-axis, e is the eccentricity. **33.** Find the coordinates of the foci. **34.** Examine the behaviour of the left-hand side of the equation $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$ for λ varying from $-\infty$ to $+\infty$ and x and y fixed, to show that the roots of th equation relative to λ satisfy the inequalities $-b^2>\lambda_1>-a^2>\lambda_2$. At $\lambda=\lambda_1$ we have a hyperbola. At $\lambda=\lambda_2$ we have an ellipse. **35.** The tangents to the conic sections are $\frac{xx_0}{a^2+\lambda_1}+\frac{yy_0}{b^2+\lambda_1}=1$, $\frac{xx_0}{a^2+\lambda_2}+\frac{yy_0}{b^2+\lambda_2}=1$. For them the orthogonality condition is $\frac{x_0^2}{(a^2+\lambda_1)(a^2+\lambda_2)}+\frac{y_0^2}{(b^2+\lambda_1)(b^2+\lambda_2)}=0$. This condition really holds. We have $\frac{x_0^2}{a^2+\lambda_1}+\frac{y_0^2}{b^2+\lambda_1}=1$, $\frac{x_0^2}{a^2+\lambda_2}+\frac{y_0^2}{b^2+\lambda_2}=1$. Subtracting these relations term by term and cancelling $(\lambda_2-\lambda_1)$, we obtain $\frac{x_0^2}{(a^2+\lambda_1)(a^2+\lambda_2)}+$

$\frac{y_0^2}{(b^2+\lambda_1)(b^2+\lambda_2)}=0$. This completes the proof. 36. $\frac{x_0}{y_0}\cdot\frac{b^2}{a^2}$. 37. Corresponding to the conjugate diameters of the ellipse are the values t_1 and t_2 of the parameter t, which differ by $\pi/2$. For a hyperbola the difference of the squares of conjugate diameters is constant. 38. Make use of the fact that the tangents at the points of intersection of the diameter with a conic section are parallel to the conjugate diameter. 39. See Exercise 38. 40. Make use of the fact that a parallelogram with its vertices at the ends of conjugate diameters is the projection of the square inscribed in the circle (see Exercise 38). 41. Make use of the fact that the ellipse is the projection of a circle. 42. Make use of the fact that the ellipse is the projection of a circle and of the properties of parallel projection. 43. Reduce the equations of the curves to canonic form (a) ellipse, (b) hyperbola, (c) parabola, (d) pair of various straight lines, (e) pair of coincident straight lines. 44. The equation can be written in equivalent form $(ax+by+c+a_1x+b_1y+c_1)(ax+by+c-a_1x-b_1y-c_1)=0$. 45. The curve is within the parallelogram determined by the intersection of two strips $|ax+by+c|\leqslant\sqrt{k}$, $|\alpha x+\beta y+\gamma|\leqslant\sqrt{k}$. 46. Take as new axes of coordinates the bisectors of the angles formed by the straight lines $ax+by+c=0$, $\alpha x+\beta y+\gamma=0$. 47. The problem can be reduced to the previous one by expanding the left-hand side of the equation into two linear factors. 48. See the hint to Exercise 49. 49. The second-degree curve $ax^2+bxy+cy^2+dx+ey=0$ admits of the parametric equation $x=-\frac{d+et}{a+bt+ct^2}$, $y=-\frac{dt+et^2}{a+bt+ct^2}$. It follows that two different second-degree curves have only four points in common.

Chapter V

1. (a) In the xy-plane lies the point D; (b) on the z-axis lies the point C; (c) in the yz-plane lies the point B. 2. $A_{xy}(1, 2, 0)$, $A_{xz}(1, 0, 3)$, $A_{yz}(0, 2, 3)$, $A_x(1, 0, 0)$, $A_y(0, 2, 0)$, $A_z(0, 0, 3)$. 3. (a) The distance to the xy-plane is 3, to the xz-plane 2, and to the yz-plane 1; (b) the distance to the x-axis is $\sqrt{13}$, to the y-axis $\sqrt{10}$, to the z-axis $\sqrt{5}$; (c) the distance to the origin of coordinates is $\sqrt{14}$. 4. $D\left(\frac{1}{4}, \frac{1}{4}, 0\right)$. 5. (2, 2, 2) and (—2, —2, —2). 6. (0, 0, 0). 7. $x+2y+3z=7$. 8. See that the diagonals of a quadrilateral intersect and are bisected by the point of intersection. 9. Show at first that the given four points are the vertices of the parallelogram. 10. $B(0, -1, 3)$. 11. $D(6, 2, -2)$, $E(3, 2, 1)$. 12. The points symmetric to (1, 2, 3) about the xy-, yz-, and xz-planes are respectively (1, 2, —3), (—1, 2, 3), (1, —2, 3). 13. (—1, —2, —3), (0, 1, —2), (—1, 0, 3). 14. $a=1$, $b=1$, $c=-2$. 15. (—1, —2, 1). 16. The equal vectors are $\overrightarrow{AB}$ and $\overrightarrow{DC}$, $\overrightarrow{BC}$ and $\overrightarrow{AD}$. 17. $D(-2, 3, 0)$. 18. $D(2, 1, -2)$. 19. $n=\frac{4}{3}$, $m=\frac{9}{2}$. 20. $\overrightarrow{AB}\left(\frac{2}{3}, \frac{1}{3}, 0\right)$. 21. $n=\frac{1}{3}$. 22. $c=1$. 23. $\sqrt{a^2+b^2+c^2+|a|\cdot|b|}$. 24. (a) $\cos\varphi=\frac{1}{\sqrt{3}}$; (b) $\varphi=90°$. 25. $\cos\varphi=\frac{2}{3\sqrt{7}}$. 26. $\cos C=\frac{2}{\sqrt{15}}$. 27. The vectors $\mathbf{a}\wedge\mathbf{b}$ and $\mathbf{c}$ are collinear. 28. The vectors $(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c}$ and $\mathbf{b}(\mathbf{ac})$ are equal in magnitude and have the same direction. 29. Represent $\mathbf{a}$ as the sum of the vectors parallel and perpendicular

to c. Next, make use of the results of Exercises 27 and 28. **30.** Make use of the results of the three previous exercises. **31.** If **a**, **b**, **c** are the vectors with the origin at the vertex of the pyramid and the ends at the vertices of its base, then $S=\frac{1}{2}[(\mathbf{a}-\mathbf{b})\wedge(\mathbf{a}-\mathbf{c})]$. Answer:

$$S=\sqrt{\left(\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\right)\left[(1-\sqrt{2})\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\right]}\times$$
$$\sqrt{\left[\sin\frac{\alpha}{2}+(1-\sqrt{2})\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\right]\cdot\left[\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+(1-\sqrt{2})\sin\frac{\gamma}{2}\right]}$$

32. Make use of the identity $(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c}=\mathbf{b}(\mathbf{ac})-\mathbf{a}(\mathbf{cd})$. **33.** Take as **a**, **b**, **c** the vectors with the origin at the centre of the sphere and the termini at the vertices of the spherical triangle. **34.** Make use of the formula of Exercise 30. **35.** Make use of the identity $(\mathbf{a}\wedge\mathbf{b})\wedge(\mathbf{c}\wedge\mathbf{d})+(\mathbf{c}\wedge\mathbf{d})\wedge(\mathbf{a}\wedge\mathbf{b})=0$. **36.** The vector **r** admits of the representation $\mathbf{r}=\lambda_1\mathbf{e}_1+\lambda_2\mathbf{e}_2+\lambda_3\mathbf{e}_3$. Find $\lambda_1, \lambda_2, \lambda_3$ by multiplying this relation scalarly by the vectors $\mathbf{e}_1\wedge\mathbf{e}_2$, $\mathbf{e}_2\wedge\mathbf{e}_3$, $\mathbf{e}_3\wedge\mathbf{e}_1$. **37.** Represent the solution as $\mathbf{r}=\lambda_1\mathbf{a}+\lambda_2\mathbf{b}+\lambda_3\mathbf{c}$. Multiplying this equality scalarly by $\mathbf{a}\wedge\mathbf{b}$, $\mathbf{b}\wedge\mathbf{c}$, $\mathbf{c}\wedge\mathbf{a}$, find $\lambda_1, \lambda_2, \lambda_3$. **38.** The vectors $\mathbf{e}_1\wedge\mathbf{e}_2$, $\mathbf{e}_2\wedge\mathbf{e}_3$, $\mathbf{e}_3\wedge\mathbf{e}_1$ are not coplanar. Therefore, $\mathbf{r}=\lambda_1(\mathbf{e}_1\wedge\mathbf{e}_2)+\lambda_2(\mathbf{e}_2\wedge\mathbf{e}_3)+\lambda_3(\mathbf{e}_3\wedge\mathbf{e}_1)$. Multiplying this relation scalarly by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ gives $\lambda_1, \lambda_2, \lambda_3$. **39.** Represent the solution in the form $x=\lambda_1(\mathbf{b}\wedge\mathbf{c})+\lambda_2(\mathbf{c}\wedge\mathbf{a})+\lambda_3(\mathbf{a}\wedge\mathbf{b})$. Multiplying this relation scalarly by a, **b**, **c** gives $\lambda_1, \lambda_2, \lambda_3$. **40.** Any three coplanar vectors are linearly related, i.e. there exist simultaneously nonzero numbers $\lambda_1, \lambda_2, \lambda_3$ such that $\lambda_1\mathbf{r}_1+\lambda_2\mathbf{r}_2+\lambda_3\mathbf{r}_3=0$. Multiplying this relation scalarly by $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ gives

$$\lambda_1(\mathbf{r}_1\mathbf{r}_1)+\lambda_2(\mathbf{r}_1\mathbf{r}_2)+\lambda_3(\mathbf{r}_1\mathbf{r}_3)=0,$$
$$\lambda_1(\mathbf{r}_2\mathbf{r}_1)+\lambda_2(\mathbf{r}_2\mathbf{r}_2)+\lambda_3(\mathbf{r}_2\mathbf{r}_3)=0,$$
$$\lambda_1(\mathbf{r}_3\mathbf{r}_1)+\lambda_2(\mathbf{r}_3\mathbf{r}_2)+\lambda_3(\mathbf{r}_3\mathbf{r}_3)=0.$$

This system of equations for $\lambda_1, \lambda_2, \lambda_3$ has a nontrivial solution (not all λ's are zero). Therefore, the determinant of the system is zero. **41.** See hint to Exercise 40. **42.** See hint to Exercise 41. **43.** See hint to Exercise 36. **44.** See hint to Exercise 38. **46.** Make use of the identity of Exercise 34. **47.** Make use of the identity of Exercise 45. **48.** $d^2=(y_1-x_1)^2+(y_2-x_2)^2+(y_3-x_3)^2+2(y_1-x_1)\times\cos\alpha+2(y_2-x_2)\cos\beta+2(y_3-x_3)\cos\gamma$. **49.** $a/2$, $b/2$, $c/2$. **50.** If (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) are the vertices of a tetrahedron, then the point of intersection of the straight lines connecting the midpoints of the opposite edges has the coordinates $\frac{x_1+x_2+x_3+x_4}{4}$, $\frac{y_1+y_2+y_3+y_4}{4}$, $\frac{z_1+z\ +z_3+z_4}{4}$. **51.** For each segment connecting the vertex of a tetrahedron with the centre of mass of the opposite face find the coordinates of the point that divides this line segment in the ratio 3:1 reckoning from the vertex. **52.** The point with the coordinates x, y, z is the centre of mass for the masses $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ at the vertices of the tetrahedron. **53.** The equation $\begin{vmatrix} x & y & z & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}=0$ is linear in x, y, z. Therefore, this is the equation of a plane. In this plane lie the points A_i, since their coordinates obey this equation. **54.** The equation admits of the equivalent representation $(x-a)^2+(y-b)^2+(z-c)^2=(\sqrt{a^2+b^2+c^2-d^2})^2$. **55.** The equation $\lambda_1 f_1+\lambda_2 f_2=0$ is the equation of a sphere. This sphere passes through the circle of intersection of these spheres, since for the points on that circle $f_1 = 0, f = 0$. By appropriately selecting λ_1 and λ_2 we can make this sphere pass through the given point. 56. If the coordinates of the point $A\ (x, y, z)$ satisfy the equation $\varphi(x, y) = 0$, then the coordinates of any point on the straight line passing through A and parallel to the z-axis satisfy this equation. 57. Let $A\ (x, y, z)$ be an arbitrary point on the cone. Then $\overrightarrow{OA}\cdot \mathbf{e}_z = |\overrightarrow{OA}| \cos\alpha$. Hence the equation of the cone is $z^2 = (x^2 + y^2 + z^2)\times \cos^2\alpha$. 58. The curves can be given by parametric equations

$$\gamma_1:\ \begin{matrix} x = u, \\ y = 0, \\ z = au^2, \end{matrix} \qquad \gamma_2:\ \begin{matrix} x = 0, \\ y = v, \\ z = bv^2. \end{matrix}$$

The coordinates of the points on the surface are

$$x = \frac{u+0}{2} = \frac{u}{2},$$
$$y = \frac{0+v}{2} = \frac{v}{2},$$
$$z = \frac{au^2 + bv^2}{2}.$$

Substituting u and v from the first two equations into the third, we obtain the equation of the surface in implicit form $z = 2ax^2 + 2by^2$. 59. We go over to the parametric equations of the curves

$$\gamma_1:\ \begin{matrix} x = t, \\ y = a, \\ z = f(t) \end{matrix} \qquad \gamma_2:\ \begin{matrix} x = t \\ y = b \\ z = \varphi(t). \end{matrix}$$

The straight line in question connects the points $(t, a, f(t))$, and $(t, b, \varphi(t))$. The coordinates of the points on this straight line can be represented in the form

$$x = \lambda t + (1 - \lambda)\, t,$$
$$y = \lambda a + (1 - \lambda)\, b,$$
$$z = \lambda f(t) + (1 - \lambda)\, \varphi(t).$$

This is the parametric equation of the desired surface (parameters t and λ). By expressing λ and t from the first two equations and substituting into the third one, we find the equation for the surface in implicit form $z = \frac{y-b}{a-b} f(x) + \frac{a-y}{a-b}\varphi(x)$. 60. We use as parameters the distance from a point on the surface to the z-axis and the angle of rotation. Then $x = r\cos\theta$, $y = r\sin\theta$, $z = f(r)$. 61. The equation $f(x) - \varphi(y) = 0$ is the equation of a cylindrical surface (see Exercise 56). It can be written as $(f(x) - z) - (\varphi(y) - z) = 0$. It is seen that the points on the curve given by the equations $z = f(x)$, $z = \varphi(x)$ satisfy this equation. 62. $x' = a_{11}x + a_{12}y + a_1$, $y' = a_{21}x + a_{22}y + a_2$, $z' = z$. 63. The equation of the sphere can be written as $(x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z)^2 = R^2$. Comparison of it with the given equation gives

$$a_{11} = \mathbf{e}_x^2,\ a_{22} = \mathbf{e}_y^2,\ a_{33} = \mathbf{e}_z^2,$$

$$a_{12} = \mathbf{e}_x\mathbf{e}_y,\ a_{\ 3} = \mathbf{e}_y\mathbf{e}_z,\ a_{31} = \mathbf{e}_z\mathbf{e}_x,$$

$$\cos\alpha = \frac{\mathbf{e}_y\mathbf{e}_z}{\sqrt{\mathbf{e}_y^2\mathbf{e}_z^2}} = \frac{a_{23}}{\sqrt{a_{22}a_{33}}}, \quad \cos\beta = \frac{a_{31}}{\sqrt{a_{33}a_{11}}}, \quad \cos\gamma = \frac{a_{12}}{\sqrt{a_{11}a_{22}}}.$$

64. Make use of the results of Exercises 43 and 44.

Chapter VI

1. $\left|\frac{d}{a}\right|, \left|\frac{d}{b}\right|, \left|\frac{d}{c}\right|$. 2. Pay attention to the fact that both equations do not contain z. Therefore, if the point (x, y, z) obeys these equations, then each point on the straight line passing through this point parallel to the z-axis obeys them. 3. The simultaneous equations $ax + by + cz + d = 0$, $ax + by + cz + d_1 = 0$ are not consistent. Subtracting the equations termwise gives $d - d_1 = 0$, which contradicts the condition of the problem. 4. See Exercise 3. By properly selecting d' we can make the plane $ax + by + cz + d' = 0$ pass through the given point. 5. The vectors $\overrightarrow{(a, b, c)}$ and $\overrightarrow{(k, l, m)}$ must be collinear. Both are perpendicular to the plane $ax + by + cz + d = 0$. 6. $kx + ly + mz = 0$. 7. $(2, 1, -2)$. 8. The simultaneous equations $x + y + z = 1$, $2x + y + 3z + 1 = 0$, and $x + 2z + 1 = 0$ are not consistent. Adding together term by term the first and the third equations and subtracting the second one gives $1 = 0$. 9. At $c = 0$. 10. Any vector $\overrightarrow{(k, l, m)}$, for which $2k + 3l + m = 0$ is parallel to the plane, e.g. $\overrightarrow{(1, -1, 1)}$. 11. Take the vector product of $\overrightarrow{(2, 3, 1)}$ and $\overrightarrow{(1, 1, 1)}$. 12. Make use of the fact that the desired plane is the locus of points equidistant from the given points. 13. The equation permits the equivalent representation $(ax + by + cz + d + \alpha x + \beta y + \gamma z + \delta)\ (ax + by + cz + d - \alpha x - \beta y - \gamma z - \delta) = 0$. It is seen that the equation defines two planes: $ax + by + cz + d \pm (\alpha x + \beta y + \gamma z + \delta) = 0$. 14. By subtracting the equations termwise we will obtain the equation of the plane $a_1x + b_1y + c_1z + d_1 - (a_2x + b_2y + c_2z + d_2) = 0$. Satisfying this equation are the points of the curve given by

$$f(x, y, z) + a_1x + b_1y + c_1z + d_1 = 0,$$
$$f(x, y, z) + a_2x + b_2y + c_2z + d_2 = 0.$$

Hence, the curve is plane. 15. $ax + by + cz + d - (\alpha x + \beta y + \gamma z + \delta) = 0$. See the hint to Exercise 14. 16. Inversion relative to the origin of coordinates is given by

$$x = \frac{R^2x'}{x'^2 + y'^2 + z'^2}, \quad y = \frac{R^2y'}{x'^2 + y'^2 + z'^2}, \quad z = \frac{R^2z'}{x'^2 + y'^2 + z'^2}.$$

18. The equation $\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$ is linear in x and y. Satisfying it are the coordinates of the given points (x_i, y_i, z_i). 19. The plane intersects the positive x-axis, if $d/a < 0$. 20. The volume of the tetrahedron is $V = \frac{1}{6}\left|\frac{d^3}{abc}\right|$. 21. The set of points in space that meet the condition $|x| + |y| + |z| < a$ is the intersection (general part) of half-spaces defined by the inequalities $\pm x \pm y \pm z < a$. This is an octahedron with vertices at the points $(\pm a, 0, 0)$, $(0, \pm a, 0)$, $(0, 0, \pm a)$. 22. The plane symmetric to the plane about the xy-plane is given by the equation $ax + by - cz + d = 0$. 23. The plane parallel to the z-axis does not contain z in its equation. Hence, the parameter λ is determined

by the condition $c + \gamma\lambda = 0$. 24. The parameters λ and μ are found from the conditions $a_1 + \lambda a_2 + \mu a_3 = 0$ and $b_1 + \lambda b_2 + \mu b_3 = 0$. 25. The distance between the planes is $\delta = \frac{|d-d'|}{\sqrt{a^2+b^2+c^2}}$. 26. $\frac{|d|}{\sqrt{a^2+b^2}}$. 27. If the planes are given by the equations in normal form $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, then the locus is given by the equations $a_1x + b_1y + c_1z + d_1 \pm \lambda(a_2x + b_2y + c_2z + d_2) = 0$. Hence it consists of two planes. 28. See Exercise 25. 29. Change to the normal form of the equation of planes. 30. See Exercise 38 in Chapter 3. 31. If the equations of planes are reduced to normal form, then

$$\pm x' = a_1x + b_1y + c_1z + d_1,$$
$$\pm y' = a_2x + b_2y + c_2z + d_2,$$
$$\pm z' = a_3x + b_3y + c_3z + d_3.$$

32. The vector $\overrightarrow{(a, b, c)}$ is perpendicular to the plane. The angle α formed by the plane with the x-axis is found from the condition $\sin\alpha = \frac{|a|}{\sqrt{a^2+b^2+c^2}}$, $\alpha \leqslant \frac{\pi}{2}$. 33. The angle formed by the given plane with the xy-plane is found from the condition $\cos\alpha = \frac{1}{1+p^2+q^2}$. 34. See Exercise 33. 35. The plane intersects the x- and y-axes under equal angles if $|a| = |b|$. 36. The parameters λ and μ must meet the condition $(\lambda a_1 + \mu a_2)a + (\lambda b_1 + \mu b_2)b + (\lambda c_1 + \mu c_2)c = 0$. 37. For any $\mathbf{n}(a, b, c)$ in the pencil of planes a plane with the normal $\mathbf{n}$ can be found. To this end, we must take $\lambda_1, \lambda_2, \lambda_3$ such that they meet the conditions $\frac{\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3}{a} = \frac{\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3}{b} = \frac{\lambda_1 c_1 + \lambda_2 c_2 + \lambda_3 c_3}{c}$. 38. The straight line intersects the x-axis (or the y-, z-axes, respectively), if $\frac{y_0}{l} = \frac{z_0}{m} \times \left(\frac{x_0}{k} = \frac{z_0}{m}, \frac{x_0}{k} = \frac{y_0}{l}\right)$. The straight line is parallel to the xy-plane (yz-, zx-planes, respectively), if $m = 0$ ($k = 0$, $l = 0$, respectively). 39. Form the equation of the locus for the normal form of the equations of the planes. 40. The locus of points equidistant from two vertices of a triangle is a plane. The desired locus is the intersection of two planes, and hence, a straight line. 41. The straight line given by the intersection of the planes $y = \lambda$, $z = a\lambda x$ lies on the surface, since the points of this curve obey the equation of the surface. The straight line given by the equations $x = \mu$, $z = a\mu y$ also lies on the surface. 42. When the determinant is zero the following system of equations is consistent:

$$a_1x + b_1y + c_1z + d_1 = 0,$$
$$a_2x + b_2y + c_2z + d_2 = 0,$$
$$a_3x + b_3y + c_3z + d_3 = 0,$$
$$a_4x + b_4y + c_4z + d_4 = 0.$$

But this system is consistent, since the straight lines intersect. 43. The vector of the parallel straight line has the coordinates $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.
44. See the hint to Exercise 43. 45. The equation of the conic surface is $\frac{|(x-x_0)a + (y-y_0)b + (z-z_0)c|^2}{a^2+b^2+c^2} = [(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]\sin^2\alpha$.

46. $\frac{x-x_0}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}=\frac{y-y_0}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}=\frac{z-z_0}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$. 47. Let A (x, y, z) be a point of the conic surface other than a vertex. We find the coordinates of the points of intersection of the generator passing through the point A with the plane $ax + by + cz + d = 0$. Substituting these coordinates into the equation of the sphere $x^2 + y^2 + z^2 = 2Rz$, we obtain the equation of the desired conic surface. The intersection of a conic surface with the xy-plane is a circle. 48. See Exercise 47. 49. If the straight lines are given by the equations $\frac{x-x'}{k'}=\frac{y-y'}{l'}=\frac{z-z'}{m'}$, $\frac{x-x''}{k''}=\frac{y-y''}{l''}=\frac{z-z''}{m''}$, then the plane that is equidistant from them passes through the point with the coordinates $\frac{x'+x''}{2}$, $\frac{y'+y''}{2}$, $\frac{z'+z''}{2}$ parallel to $\overrightarrow{(k', l', m')}$, $\overrightarrow{(k'', l'', m'')}$. 50. The plane given by

$$\frac{a_1x+b_1y+c_1z+d_1}{a_1x_0+b_1y_0+c_1z_0+d_1}=\frac{a_2x+b_2y+c_2z+d_2}{a_2x_0+b_2y_0+c_2z_0+d_2},$$

passes through the given point and a point (x_0, y_0, z_0) that does not lie on the straight line. 51. The vector $\overrightarrow{(x'-x_0, y'-y_0, z'-z_0)\wedge(k, l, m)}$ is perpendicular to the desired plane. 52. Any straight line which meets the two given straight lines can be represented as an intersection of two planes, one of which passes through the first line, the other through the second. 53. The surface given by equations of the form $\varphi\left(\frac{x}{z}, \frac{y}{z}\right)=0$ is formed by the straight lines passing through the origin of coordinates, since together with point (x, y, z) the equation is satisfied by any point $(\lambda x, \lambda y, \lambda z)$. The surface intersects the plane $z = 1$ along the curve $\varphi(x, y) = 0$.

Chapter VII

1. The surface $z = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a$ is an elliptical paraboloid (hyperbolic paraboloid, parabolic cylinder). 2. The left-hand side of the equation can be represented as the product of two linear factors. 3. The coordinates of points on the curve along which the plane intersect the surface satisfy the equation which results. 4. See Exercise 3. 5. Form the equation of the conic surface. Take this point as the origin of coordinates, and the plane, in which curve lies as the plane z = const (see Exercise 57, Ch. V). 6. The quadric surface $x^2 + y^2=\left(\frac{z-b}{a}\right)^2+\left(\frac{z-d}{c}\right)^2$. 7. The foci are on the z-axis at the distance $\sqrt{c^2-a^2}$ from the origin of coordinates. 8. The intersection of the ellipsoid with the planes is at the same time the intersection of these planes with the sphere $x^2 + y^2 + z^2 + \mu = 0$. 9. Eliminate the parameters u, v and change to the equation of the surface in implicit form. 10. Ellipsoid. To prove this use the bounded nature of the surface. 11. See Exercise 8. 12. See the hint to Exercise 34, Ch. IV. 13. Consider the projection of the line of intersection on the xy-plane. 14. The first family is $x = \lambda$, $z = a\lambda y$, the second family is $y = \mu$, $z=a\mu x$. 16. To use the fact that $\overrightarrow{(\lambda, \mu, \nu)}$ and $\overrightarrow{(x, y, z)}$ form the angle α. 17. If $\overline{A}$ is the projection of the point A (x, y, z) on the straight line $\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}$, then $(O\overline{A})^2+R^2=(OA)^2$. Exp-ess $|O\overline{A}|$ in terms of the scalar product of $\overrightarrow{(\lambda, \mu, \nu)}$

and $\overrightarrow{(x, y, z)}$. 18. See Exercise 16. 19. The diameters of the parabola are parallel to the straight line $ax + by + c = 0$. The axis of the parabola is the straight line $ax + by + c + \frac{a\alpha + b\beta}{2(a^2 + b^2)} = 0$.

Chapter VIII

1. $x = a \cos \omega t$, $y = a \sin \omega t$, $z = ct$. 2. $x = vt - a \sin \frac{vt}{a}$, $y = a - a \cos \frac{vt}{a}$. 3. $x = \frac{3at}{1+t^3}$, $y = \frac{3at^2}{1+t^3}$. 4. If the pencil of the projecting lines is parallel to the yz-plane, then the equations of the projection are $x = a \cos \omega t$, $y = ct \tan \theta + a \sin \omega t$. The projection has singular points if $\tan \theta = \pm \frac{a\omega}{c}$. They are cusps of the first kind. 5. Singular points are cusps of the first kind. 6. The singular points are $(\pm a, 0)$, $(0, \pm a)$. They are cusps of the first kind. 7. The singular point $(a, 0)$ is a cusp of the first kind. 8. The equation of the tangent is $\frac{x-1}{0} = \frac{y}{1} = \frac{z}{1}$, of the osculating plane $y - z = 0$, of the normal plane $y + z = 0$, of the principal normal $y = z = 0$, and of the binormal $\frac{x-1}{0} = \frac{y}{1} = \frac{z}{-1}$. 9. $\frac{x}{0} = \frac{y}{1} = \frac{z-1}{0}$. 10. $x = 0$. 11. $y = x^2 \pm 3x + 3$. 12. Find the length of the tangent segment. 13. A helix. Find its equation, taking that from Ex. 1. 14. At right angles. The tangents to the curves at their common point (x, y) are perpendicular. 15. See the hint to Ex. 34, Ch. IV. 16. The equation of the tangent at an arbitrary point t of the curve is $\frac{x - x(t)}{x'(t)} = \frac{y - y(t)}{y'(t)} = \frac{z - z(t)}{z'(t)}$. Without loss of generality, we can assume that the tangents pass through the origin. Then $\frac{x(t)}{x'(t)} = \frac{y(t)}{y'(t)} = \frac{z(t)}{z'(t)}$. Hence, $y'x - x'y = 0$, and $\left(\frac{y}{x}\right)' = 0$, i.e., $\frac{y}{x} = c_1 = \text{const}$. Similarly, $\frac{z}{x} = c_2 = \text{const}$. Thus, our curve is at the intersection of the two planes, which means that it is either a straight line or its part. 17. Find the angle θ between the tangent and the z-axis. Find the equation of the principal normal at an arbitrary point, and show that it is the equation of a straight line cutting the z-axis. 18. Let $\mathbf{n}(a, b, c)$ be a vector perpendicular to the plane. The tangent vector of the curve is perpendicular to $\mathbf{n}$. Hence, $ax'(t) + by'(t) + cz'(t) = 0$, which means that $ax(t) + by(t) + cz(t) = d = \text{const.}$, i.e., the curve is in the plane $ax + by + cz = d$. 19. The point of the curve $(x(t), y(t), z(t))$ satisfies the two pairs of simultaneous equations identically with respect to t:

$$\begin{aligned} a_1(t)\, x(t) + b_1(t)\, y(t) + c_1(t)\, z(t) + d_1(t) &= 0 \\ a_2(t)\, x(t) + b_2(t)\, y(t) + c_2(t)\, z(t) + d_2(t) &= 0 \end{aligned} \qquad (*)$$

$$\begin{aligned} a_1(t)\, x'(t) + b_1(t)\, y'(t) + c_1(t)\, z'(t) &= 0 \\ a_2(t)\, x'(t) + b_2(t)\, y'(t) + c_2(t)\, z'(t) &= 0 \end{aligned} \qquad (**)$$

The first two mean that the point of the curve belongs to the tangent, and the other that the tangent vector is parallel to the planes whose intersection forms the tangent. Differentiating the first two equations with respect to t, we obtain by means of the latter two that

$$\left.\begin{array}{l} a_1'(t)\,x(t)+b_1'(t)\,y(t)+c_1'(t)\,z(t)+d_1'(t)=0 \\ a_2'(t)\,x(t)+b_2'(t)\,y(t)+c_2'(t)\,z(t)+d_2'(t)=0 \end{array}\right. . \qquad (***)$$

Thus, for the functions $x(t)$, $y(t)$, $z(t)$, we have the four equations (*) and (***) from which they can be found. Anyway, in order that they may be solved, it is necessary that

$$\begin{vmatrix} a_1(t) & b_1(t) & c_1(t) & d_1(t) \\ a_2(t) & b_2(t) & c_2(t) & d_2(t) \\ a_1'(t) & b_1'(t) & c_1'(t) & d_1'(t) \\ a_2'(t) & b_2'(t) & c_2'(t) & d_2'(t) \end{vmatrix} = 0.$$

20. Let $\begin{vmatrix} \varphi_y & \psi_y \\ \varphi_z & \psi_z \end{vmatrix} \neq 0$ at the point (x, y, z) in question. Then the curve is given by the equations $y = y(x)$, $z = z(x)$ in the neighbourhood of the point. Differentiating the identities $\varphi(x, y(x), z(x)) = 0$, $\psi(x, y(x), z(x)) = 0$, we consecutively find $y'(x)$, $z'(x)$, $y''(x)$, $z''(x)$, and then easily write down the osculating plane equation. 21. Apply the argument used in the hint to Ex. 19. 22. The family of the straight lines cutting off a triangle of area $\frac{a^2}{2}$ can be given by the equation $\frac{x}{\lambda}+\lambda y=a$, λ being the parameter. The envelope of the family is the branch of the hyperbola $xy=\frac{a^2}{4}$ inside the angle Oxy. 23. Find the implicit equation of the path of a point mass projected with velocity v_0 at an angle α to the horizontal, and then find the envelope of the paths. Answer: $y=-\frac{gx^2}{2v_0}+\frac{v_0^2}{2g}$, g is the acceleration due to gravity.

Chapter IX

1. $s=\frac{2ab\sqrt{1+4a^2b^2}+\ln(2ab+\sqrt{1+4a^2b^2})}{2b}$. 2. $s=a\sqrt{2}\sinh t$. 3. $s=ba$. 4. $s = 8a$. 5. $s=\int_{\theta_1}^{\theta_2}\sqrt{\rho^2+\rho'^2}\,d\theta$. 6. $k_1=\frac{1}{4}\sqrt{1+\sin^2\frac{t}{2}}$. 7. The curve can be given by equations of the form $y = y(x)$, $z = z(x)$ in the neighbourhood of the point in question. Find the derivatives y', y'', z' and z'' for $x = 0$. Curvature can then be found easily. Answer: $k_1=\frac{\sqrt{6}}{9}$. 8. Make use of the parametric equation of the circle $x=R\cos t$, $y=R\sin t$. Answer: $k_1=\frac{1}{R}$. 9. $k_1=\frac{1}{2a\cosh^2 t}$, $k_2=\frac{1}{2a\cosh^2 t}$. 10. The ellipse is given by the equation $y=b\sqrt{1-\frac{x^2}{a^2}}$ in the neighbourhood of the vertex $(0, b)$. We obtain $k_1=\frac{b}{a^2}$ for the curvature at this vertex, which is the same at $(0, -b)$. The curvature is $k_1=\frac{a}{b^2}$ on the x-axis. 11. Find the curvature and torsion of the helix, and show that they do not depend

on parameter. 12. Apply the general formula for the curvature of a curve given by equations $x = \rho(\theta)\cos\theta$, $y = \rho(\theta)\sin\theta$. 13. Let **a** be the direction vector of the straight line, and $\boldsymbol{\tau}$ the unit tangent vector of the curve. We have $\mathbf{a}\boldsymbol{\tau} =$ const. Differentiating with respect to the arc of the curve, and noticing that $\boldsymbol{\tau}' = k\mathbf{v}$, we obtain $\mathbf{a}\mathbf{v} = 0$, i.e., the principal normal is perpendicular to the straight line. 14. The osculating plane equation can be written in the form $(\mathbf{r} - \mathbf{r}(s))\boldsymbol{\beta} = 0$, where $\boldsymbol{\beta}$ is the unit binormal vector. Without loss of generality, we can assume that the osculating planes pass through the origin. Then $\mathbf{r}(s)\boldsymbol{\beta}(s) \equiv 0$. Differentiating with respect to s, we obtain $\boldsymbol{\tau}\boldsymbol{\beta} + \mathbf{r}(k_2\boldsymbol{\tau}) = k_2\mathbf{r}\boldsymbol{\tau} \equiv 0$. If the tangent to the curve does not pass through the origin, then $\mathbf{r}\boldsymbol{\tau} \neq 0$. Therefore, $k_2 = 0$, and the curve is plane. (If, for any s, the tangent does pass through the origin, then the curve is either a straight line or its part.) 15. $k_2 = 1$. 16. From the data, $\mathbf{a}\boldsymbol{\tau} =$ const, where **a** is a constant vector, and $\boldsymbol{\tau}$ the tangent unit vector. Differentiating with respect to the arc s, we obtain $\mathbf{a}k_1\mathbf{v} = 0$. For $k_1 \neq 0$, $\mathbf{a}\mathbf{v} = 0$. Diffferentiating with respect to s again, we obtain $-\mathbf{a}k_1\boldsymbol{\tau} - \mathbf{a}k_2\boldsymbol{\beta} = 0$. Since the vector $\boldsymbol{\tau}$ makes a constant angle with **a**, and $\mathbf{v}$ is perpendicular to **a**, $\boldsymbol{\beta}$ also forms a constant angle with **a**. Therefore, $\mathbf{a}\boldsymbol{\beta} =$ const, and it follows from $k_1(\mathbf{a}\boldsymbol{\tau}) + k_2(\mathbf{a}\boldsymbol{\beta}) = 0$ that $\frac{k_1}{k_2}$ is constant. 17. A semicubical parabola $27py^2 = 8(x-p)^3$. 19. $x = R(\cos\theta + (\theta - c)\sin\theta)$, $y = R(\sin\theta - (\theta - c)\cos\theta)$. 20. $x = \int \sin\alpha(s)\,ds$, $y = \int \cos\alpha(s)\,ds$, where $\alpha(s) = \int k(s)\,ds$. 21. Assume that the function $\boldsymbol{\tau}(s)$ is given. We have $\boldsymbol{\tau}(s) = \mathbf{r}'(s)$. Hence, $\mathbf{r}(s) = \int \boldsymbol{\tau}(s)\,ds$. If either $\boldsymbol{\beta}(s)$ or $\mathbf{v}(s)$ is given, then we first find $\boldsymbol{\tau}(s)$. We have $\boldsymbol{\beta}' = k_2\boldsymbol{\tau}$. Hence, $\boldsymbol{\tau} = \frac{\boldsymbol{\beta}'}{|\boldsymbol{\beta}|}$. Now, $\mathbf{r}(s) = \int \frac{\boldsymbol{\beta}'(s)}{|\boldsymbol{\beta}(s)|}\,ds$. Let $\mathbf{v}(s)$ be given. We have $\mathbf{v}' = -k_1\boldsymbol{\tau} - k_2\boldsymbol{\beta}$. Multiplying vectorially by $\mathbf{v}$, we obtain $\mathbf{v}' \wedge \mathbf{v} = -k_1\boldsymbol{\beta} + k_2\boldsymbol{\tau}$. We find $\boldsymbol{\tau}(s)$ from the two equations, and express $\mathbf{r}(s)$ in terms of it. 22. Proof is based on the use of the Frenet formulas. E.g., if the first condition is fulfilled, then $\mathbf{a}\boldsymbol{\tau} =$ const, where **a** is a constant vector. It follows that $\mathbf{a}\mathbf{v} = 0$ (see Ex. 13), and the principal normals are parallel to a plane perpendicular to the vector **a**. Further, we conclude that $\boldsymbol{\beta}\mathbf{a} =$ const, and $\frac{k_1}{k_2} =$ const (see the answer to Ex. 16). 23. The curvature and torsion of a helix are constant, and may assume any values for a convenient choice of the curve parameters. Since a curve is uniquely determined by specifying its curvature and torsion, any curve with constant curvature and torsion is a helix.

Chapter X

1. $z^2 + (\sqrt{x^2 + y^2} - a)^2 = R^2$. 2. The sphere $x^2 + y^2 + z^2 = a^2$. 3. $x = \varphi(u)\cos v$, $y = \varphi(u)\sin v$, $z = \psi(u)$. 4. $x = v\cos\omega u$, $y = v\sin\omega u$, $z = au$. 5. In moving along a helix, the principal normal rotates uniformly about its axis, and intersects it at right angles. Therefore, the surface formed by the principal normals to a helix is a helicoid (see Ex. 4). 6. For $u =$ const, the curve $\mathbf{r} = \boldsymbol{\varphi}(u) + \boldsymbol{\psi}(v)$ is obtained from the curve $\mathbf{r} = \boldsymbol{\psi}(v)$ by a translation through the vector $\boldsymbol{\varphi}(u)$. 7. If the curves are given by equations $\mathbf{r} = \mathbf{r}_1(u)$, $\mathbf{r} = \mathbf{r}_2(v)$, then the locus of the mid-points of line segments with ends on the given curves is given by $\mathbf{r} = \frac{\mathbf{r}_1(u) + \mathbf{r}_2(v)}{2}$. 8. The equation of the surface is $\mathbf{r} = \mathbf{r}(u) + v\mathbf{a}$ with parameters u and v. 9. The equation of the surface is $\mathbf{r} = \mathbf{p} - (\mathbf{r}(u) - \mathbf{p})v$, where $\mathbf{p}$ is the vector $\overrightarrow{(a, b, c)}$. 10. The equation of a curve intersecting the straight

lines is given by an equation $\mathbf{r} = \mathbf{f}(u)$. If $\varphi(u)$ is the director vector, then the vector of any point on the surface can be specified by $\mathbf{r} = \mathbf{f}(u) + v\varphi(u)$. **11.** See the derivation of the equation of tangents to an ellipse and a hyperbola in Ch. 4, Sec. 6. **12.** $x = a$. **13.** Find the equation of a tangent plane at an arbitrary point of the surface, and show that the point (0, 0, 0) satisfies it. **16.** A hyperbolic paraboloid. **17.** The equation of the ellipsoid in the neighbourhood of the point $(0, 0, c)$ can be represented in the form $z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$. The osculating paraboloid equation is $z = c + c\left(1 - \frac{1}{2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\right)$. **18.** The points are elliptic on the ellipsoid, hyperbolic on the hyperboloid, elliptic on the elliptic paraboloid, hyperbolic on the hyperbolic paraboloid, and parabolic on the cylinders and cone. **19.** Let the plane in question have the vector **a** as its normal, and the considered point as the origin. Since the surface and the plane have only one point in common, the surface is on one side of the plane. Therefore, either $\mathbf{ar}(u, v) \geqslant 0$ or $\mathbf{ar}(u, v) \leqslant 0$, equality occurring only at one point. It follows that $\mathbf{ar}_u = 0$, and $\mathbf{ar}_v = 0$ at this point, i.e., the plane is tangent to the surface. **20.** Take some point on the line as the origin, and the tangent plane to the surface at the point as the xy-plane. Represent the surface equation in the form $z = \frac{1}{2}(rx^2 + 2sxy + ty^2) + \varepsilon(x, y)(x^2 + y^2)$. At the elliptic and hyperbolic points, $rt - s \neq 0$. Hence, deduce that $z_x^2 + z_y^2 > 0$ for sufficiently small $x^2 + y^2$ if $x^2 + y^2 \neq 0$. It means that the xy-plane cannot be tangent at the points near the origin, which is contrary to the conditions of the problem. **21.** Take the tangent plane at P as the xy-plane, and represent the equation of the surface in the form $z = \frac{1}{2}(rx^2 + 2sxy + ty^2) + \varepsilon(x, y)(x^2 + y^2)$. Mind that $rx^2 + 2sxy + ty^2$ has constant sign at an elliptic point P, and is alternating at a hyperbolic one. **22.** Representing the surface equation in the form $z = z(x, y)$, we notice that $d^2z \equiv 0$ at planar points. Therefore, $d^2z \equiv 0$ along γ. It follows that $z = ax + by + c$ along γ, where a, b, c are constant, i.e., γ is planar. **23.** If we take a sphere containing a surface, and decrease its radius, then it will eventually touch the surface. The point of tangency is elliptic. **24.** Let the intersections of the surface with the planes passing through the given straight line, and the planes perpendicular to the straight line, be coordinate lines. **25.** If the point in question is the origin, then the vector of the point on the surface $\mathbf{r}(u, v)$ is a normal. Therefore, $\mathbf{rr}_u = 0$, $\mathbf{rr}_v = 0$, i.e., $\mathbf{r}\,d\mathbf{r} = 0$; consequently, $\mathbf{r}^2 = \text{const}$ (sphere).

Chapter XI

1. $(\varphi'^2 + \psi'^2)\,du^2 + \varphi^2 dv^2$. **2.** Use the result of Ex. 1. Introduce a new parameter u_1 instead of u, setting $du_1 = \sqrt{\varphi'^2 + \psi'^2}\,du$. **3.** $s = |\sinh u_2 - \sinh u_1|$. **4.** $\cos\theta = \frac{a^2 x_0 y_0}{\sqrt{1 + a^2 x_0^2}\sqrt{1 + a^2 y_0^2}}$. **5.** Find the first fundamental form, and show that $F = 0$. **6.** Let $x = R\cos u\cos v$, $y = R\cos u\sin v$, $z = R\sin u$ be the parametric equation of the sphere, and the lines $v = \text{const}$, meridians. The linear element of the sphere is $ds^2 = R^2(du^2 + \cos^2 u\,dv^2)$. Let the loxodrome $u = u(v)$ intersect the meridians at an angle θ. Find $\cos\theta = \frac{1}{\sqrt{1 + \cos^2 u \cdot u'^2}}$. Hence, $\cos u' = \tan\theta$, $\sin u = v\tan + \text{const}$ is the loxodrome equation. **7.** $s = \frac{b^2}{a}\left(\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right)$. **8.** The tangent planes of the paraboloids make equal angles with the xy-plane at the corresponding points (with the same projection). 9. The fundamental form $E\,du^2 + 2\,F du\,dv + G dv^2$ with constant coefficients is transformed into $du_1^2 + du_2^2$. Hence, the surface is locally isometric to the plane. 10. The linear element of the plane with respect to polar coordinates ρ, θ is of the form $d\rho^2 + \rho^2 d\theta^2$ (show!). Reduce the linear element of the surface of revolution $du^2 + G(u)\,dv^2$ to the form $f(u_1)(du_1^2 + u_1^2\,dv^2)$ by introducing $u = \varphi(u_1)$ instead of u. 11. Transform the linear element of the sphere $du^2 + \cos^2 u\,dv^2$ to the form $\lambda(u_1)(du_1^2 + dv^2)$. 12. Show that the linear elements of the surfaces coincide with a convenient choice of parameters. 13. $\frac{-2\,du\,dv}{\sqrt{1+u^2}}$. 14. $k_n = \frac{a\,dx^2 + b\,dy^2}{dx^2+dy^2}$. 15. The normal curvature of the surface is $k_n = \frac{L\,du^2+2M\,du\,dv+N\,dv^2}{E\,du^2+2F\,du\,dv+G\,dv^2}$; $k_n=0$ for the plane. Putting $dv=0$, we obtain $E = 0$ from the formula for k_n; putting $du = 0$, we obtain $G = 0$. Now, setting $du = dv \neq 0$, we obtain $F = 0$. k_n does not depend on the ratio $du : dv$ for the sphere. Putting $du=0$ we obtain $k_n = \frac{L}{E}$; putting $dv=0$, we obtain $k_n = \frac{N}{G}$. Set $du=dv$. Then $k_n = \frac{L+2M+N}{E+2F+G} = \frac{L}{E} = \frac{N}{G}$. Hence, $\frac{L}{E} = \frac{M}{F} = \frac{N}{G}$, i.e., the second fundamental form is proportional to the first. 16. $x = c_1 y$, $\frac{1}{x^2} - \frac{1}{y^2} = c_2$ (c_1, c_2 being constant). 17. $x = \cosh u \cos v$, $y = \cosh u \sin v$, $z = u$. 18. Find the asymptotic lines on the helicoid. 19. $M = (\mathbf{r}_{uv}\mathbf{n}) = 0$, since $\mathbf{r}_{uv} = 0$. 20. Take meridians and parallels as coordinate lines, and show that $F=0$, $M=0$. 21. a and $-a$. 22. $\ln\left(u+\sqrt{u^2+c^2}\right) - v = \text{const}$, $\ln\left(u+\sqrt{u^2+c^2}\right) + v = \text{const}$. 23. Mean curvature is zero, whereas Gaussian curvature $-a^2$. 24. Take the tangent plane to the surface as the xy-plane. 25. Find the mean curvatures of the helicoid and catenoid. 26. Find the Gaussian curvature of the cylindric surface, using the result of Ex. 8, Ch. X. 27. The equation of the surface formed by the tangents to the curve $\mathbf{r} = \mathbf{r}(u)$ is $\mathbf{r} = \mathbf{r}(u) + v\mathbf{r}'(u)$. Find the Gaussian curvature of the surface. 28. If the asymptotic lines are taken as the coordinate ones, then the second quadratic form is $2M\,dudv$ and mean curvature $H = \frac{MF}{(EG-F^2)^{3/2}}$. If $H=0$, then $F=0$, i.e., the coordinate lines are orthogonal. 29. By the Rodrigues theorem, $\mathbf{n}_u = \lambda \mathbf{r}_u$, $\mathbf{n}_v = \lambda \mathbf{r}_v$. Differentiating the first equality with respect to v, the second with respect to u, and subtracting termwise, we obtain $\lambda_v \mathbf{r}_u - \lambda_u \mathbf{r}_v = 0$. Hence $\lambda_u = 0$, $\lambda_v = 0$, and, therefore, $\lambda = \text{const}$. Integrating $d\mathbf{n} = \lambda d\mathbf{r}$, we obtain $\mathbf{n} = \lambda \mathbf{r} + c$, $(\lambda \mathbf{r} + c)^2 = 1$, which is a sphere. 30. Take the vector equation of the surface Φ in the form $\mathbf{r} = \mathbf{r}(u, v) + \lambda \mathbf{n}(u, v)$, where $\mathbf{r}(u, v)$ is the vector of a point on the surface F, and $\mathbf{n}(u, v)$ the unit normal vector at the point. Prove by the Rodrigues theorem that the tangent planes to F and Φ are parallel at the corresponding points, and also that the principal directions at the points are corresponding. 31. Prove that, along the corresponding principal directions of the surfaces F and Φ, their normal curvatures are related by $\frac{1}{k_n(\lambda)} = \frac{1}{k_n} + \lambda$, and then express the mean and Gaussian curvature of Φ in terms of those of F. 32. If the lines of curvature are taken as the coordinate lines, then, by the Rodrigues theorem, $\mathbf{n}_u = -k_1 \mathbf{r}_u$, $\mathbf{n}_v = -k_2 \mathbf{r}_v$. Since $k_1 + k_1^2 = 0$, $\mathbf{n}_u^2 = k_1^2 \mathbf{r}_u^2$, $\mathbf{n}_v^2 = k_1^2 \mathbf{r}_1^2$. Besides, since $\mathbf{r}_u \mathbf{r}_v = 0$, and $\mathbf{n}_u \mathbf{n}_v = 0$, $d\mathbf{n}^2 = k_1^2 d\mathbf{r}^2$, which just means that a spherical mapping of the surface is conformal.

Chapter XII

1. Use the Gauss formula to express Gaussian curvature in terms of the first fundamental form coefficients. 4. Take the lines of curvature on the surface as the coordinate lines. Making use of the fact that L depends only on u, and N only on v (see Ex. 3), reduce the second fundamental form to $du^2 - dv^2$. The second fundamental form is then reduced to $\lambda\ (du^2 + dv^2)$, since mean curvature is zero. 5. For an asymptotic line, the osculating plane coincides with the tangent plane to the surface, and, for a geodesic line, it is perpendicular to the tangent plane. Hence, the curvature is zero, and the curve a straight line. 6. Take as the parameter on the curve its arc. Since it is a line of curvature, $\mathbf{r}' = \lambda\mathbf{n}'$. Because it is a geodesic, $\mathbf{r}'' = \mu\mathbf{n}$. Hence, $\mathbf{r}''' = \mu'\mathbf{n} + \mu\mathbf{n}'$ $(\mathbf{r}'''\mathbf{r}''\mathbf{r}') = (\mu'\mathbf{n} + \mu\mathbf{n}'\mu\mathbf{n}\lambda\mathbf{n}') = 0$. Therefore, the torsion of the curve is zero, and the curve is plane. 7. Cylindrical surface is locally isometric to the plane. The rectilinear generators on the plane correspond to parallel straight lines. But, a straight line meets the family of parallel straight lines at the same angle. 8. The given linear element is that of the Lobachevsky plane in the Poincaré model. Therefore, geodesics are the curves $u =$ const and $(u - c_1)^2 + v^2 = c^2$. 9. Prove that all these surfaces are of zero Gaussian curvature. 10-12. Make use of the Gauss-Bonnet theorem. 13. It follows from the definition of total curvature in the sense of Gauss that if Gaussian curvature does not change sign in a domain G, then $\omega\ (G) = \left|\int\int K ds\right|$, where $\omega\ (G)$ is the area of the spherical image of G. With this in mind, prove that the Euler characteristic of a torus is zero. 14. Take a solid in the form of a cylinder, and make n circular openings in it, parallel to the axis. Smooth the surface of the obtained solid, and apply the Gauss-Bonnent theorem on the basis of the argument given in the hint to Ex. 13. Answer: $2\ (1 - n)$.

Chapter XVI

1. Complete Euclidean space with the elements at infinity, and apply the Desargues theorem. 2. Complete Euclidean space with the elements at infinity, and make use of the Desargues theorem. 3. The homogeneous coordinates of the point at infinity on the straight line are k, l, m, 0. 4. The third point coordinates are linearly expressed in terms of those of the first two as $\lambda a_1 + \mu b_1 = c_1$, $\lambda a_2 + \mu b_2 = c_2$, $\lambda a_3 + \mu b_3 = x_3$, $\lambda a_4 + \mu b_4 = x_4$. From the first two equations, we find λ and μ, and then x_3 and x_4. 5. Make use of two projections. 6. Transform the points A, B, C, D by a projective transformation into the points $A_1\ (-1, 0, 0, 1)$, $B_1\ (0, 0, 0, 1)$, $C_1\ (1, 0, 0, 1)$, $D_1\ (\xi, 0, 0, 1)$. 7. Take the cross ratio of the four points at which the given straight lines intersect the straight line $x = 1$, $\frac{\sin(\alpha-\gamma)}{\sin(\beta-\gamma)} \div \frac{\sin(\alpha-\delta)}{\sin(\beta-\delta)}$. 8. $\chi = 1$. 9. Make use of the Steiner theorem. 10. Take two points on the polar, and construct their polars; the required pole is the intersection. 11. $a_{13} = 0$, $a_{23} = 0$. 12. $a_{ij} = 0$ for $i \neq j$. 13. If a projectivity is set between the points of the two given straight lines, not reduced to simply projecting one straight line onto the other, then the straight lines joining the points touch a curve of the second degree. 14. Under polar reciprocation, the vertices are transformed into the face planes, and the face planes into the vertices. Therefore, the cube is transformed into the octahedron, and the dodecahedron into the icosahedron. 15. Use the Klein model of Lobachevskian geometry. Take the point P as the centre of the circle. Then the parallel angle is simply Euclidean. Find the distance PQ in the sense of Lobachevsky, expressing it in terms of the parallel angle. Use the distance formula. 16. Use the distance formula in the Klein model of Lobachevskian geometry, and also the condition for two straight lines to be perpendicular.

Chapter XVII

1. Use Locus 1. 2. Use Locus 5. 3. Use Locus 5. 4. The circle passing through the centres of the given circle is concentric with the required. 5. See Ex. 2. 6. The difference between the squares of the distances from the centre of the required circle to those of the two given circles equals the difference between their radii squared. Use Locus 8. 7. The ratio of the distances from the required point to the centres of the two given circles is equal to the ratio of the radii. Use Locus 6. 8. Cut off a line segment $AD = l$ on the half-line AC. The required point X is equidistant from B and D. Use Locus 3. 9. First, construct a line segment with ends on the circles, visible at their centre at the angle α, for which take any equal angle with vertex at the centre. 10. The required straight line is parallel to the diagonal of the parallelogram obtained by the given intersecting straight lines. 11. If we take chords of given length in the given circles, and construct concentric circles touching them, then the required straight line is the common tangent to the constructed circles. 12. Provided the three vertices of the parallelogram are on the sides of the quadrilateral, and the sides are of the given directions, find the locus of the fourth point (straight line). 13. Use the similarity method. 14. Use the similarity method, first constructing any square whose two vertices are on one side of a triangle, and the third on the other. 15. Apply the similarity method, first constructing some line segment parallel to the chord joining the radii ends, which is trisected by them. 16. Apply the similarity method, first constructing some rhombus whose sides are parallel to the diagonals of the quadrilateral and two adjacent vertices are on the adjacent sides of the quadrilateral 17. Use the theorem on the segments of a secant and tangent to a circle, drawn from one point. 18. Apply the similarity method, the altitudes being inversely proportional to the sides. 19. First, construct a right triangle in which the given angle bisector is the hypotenuse, and the altitude one of the sides containing the right angle. 20. First, construct a right triangle in which the given median and altitude are the hypotenuse and one of the sides about the right angle, respectively. Then find the circumcentre. 21. First, construct a right triangle in which the given side is the hypotenuse, and the given altitude one of the sides about the right angle. 22. Let ABC be the required triangle with the given angle α at the vertex C, side AB and the sum of the sides AC and BC. Cut off the line segment $AD = AC + BC$ on the half-line AC. The triangle ADB can be constructed easily, because $\angle D = \frac{\alpha}{2}$. 23. Mind that the angles of the triangle whose two vertices are the ends of the chords, and the third is the second point where the circles meet, do not depend on the straight line which should be perpendicular to the common chord. 24. If the given angle is at a circumference of given radius, then we obtain the opposite side of the required triangle, and the problem is reduced to Ex. 22. 25. Apply the translation method. Form a triangle by translating the medians. 26. If $ABCD$ is the required parallelogram, and E the point where its diagonals meet, then two sides AE and BE and the included angle of the triangle ABE are known. 27. The locus of the vertices of the required triangle is a circle, and the problem is reduced to Ex. 23. 28. Apply the translation method. 29. Apply reflection in g. 30. First, find the point D' on the straight line AC, symmetric to D about the straight line BX. 31. Find the point B' symmetric to B about g. The point X is at the intersection of the straight lines AB' and g. 32. Rotate the square about the given vertex of the triangle through 90°. 33. Apply inversion to transform the given circle into a straight line. The problem is then reduced to Ex. 17.

Chapter XIX

1. The centroid is the projection of the circumcentre of the hexagon. 2. The projection is the conjugate diameter. 3. Construct the projections of the diagonals (see (Ex. 2)). 4. First, construct the projection of the inscribed square.

5. Let A be the given vertex, O projection of the centre of the circle, and A' point symmetric to A with respect to O. The opposite side of the triangle passes through the mid-point of the line segment OA', and is parallel to the diameter conjugate to AA'. 6. Construct the projection of the inscribed regular triangle, and draw through its vertices straight lines parallel to the opposite sides.

Chapter XX

1. $\cos^{-1}\dfrac{1}{\tan\alpha\tan\frac{\pi}{n}}$. 2. The vertex angle is $\pi-2\alpha$, and the base angle $\alpha=\tan^{-1}\dfrac{1}{\cos\beta\tan\frac{\pi}{n}}$. 3. $\tan^{-1}\dfrac{\tan\gamma}{\cos\frac{\pi}{n}}$. 4. $\pi-\cos^{-1}\left(\cos^2\beta+\sin^2\beta\cos\frac{2\pi}{n}\right)$. 5. $\sin^{-1}\sqrt{\dfrac{1+\cos\delta}{1-\cos\frac{2\pi}{n}}}$. 6. $\pi-\cos^{-1}\left(\dfrac{1-\cos\frac{2\pi}{n}}{\left(\cot\frac{\varphi}{2}\cdot\tan\frac{\pi}{n}\right)^2}+\cos\frac{2\pi}{n}\right)$. 7. If α, β, γ are the face angles of the trihedral angle, then the angle between the plane of γ and the opposite edge is $\cos^{-1}\dfrac{\sqrt{\cos^2\alpha+\cos^2\beta+2\cos\alpha\cos\beta\cos\gamma}}{\sin\gamma}$. Given the dihedral angles A, B, C of given trihedral angle, the angle between the edge with C and the opposite face is $\cos^{-1}\dfrac{\sqrt{\cos^2 A+\cos^2 B+2\cos A\cos B\cos C}}{\sin C}$. 8. By a translation, send the vertex with the known angles into any other vertex at which the dihedral angles should be found. 9. Let ABC be the triangle in the prism base, and AD a lateral edge. The required angles at the vertices B and C are found by means of the scalar products $\vec{AD}\cdot\vec{BC}$ and $\vec{AD}\cdot\vec{CB}$. Make use of the decomposition $\vec{BC}=\vec{BA}+\vec{AC}$. 10. The cosines of the dihedral angles equal 1/3 for the tetrahedron, $-1/3$ for the octahedron, $-\cos\frac{2\pi}{5}\Big/2\sin^2\frac{\pi}{5}$ for the dodecaherdon, and $-\left(1+4\cos\frac{2\pi}{5}\right)\Big/3$ for the icosahedron. 11. If a is the edge of a regular polyhedron, 2γ the dihedral angle at its edges, and n the number of the sides in one face, then the inradius and circumradius are $\dfrac{a\tan\gamma}{2\tan\frac{\pi}{n}}$, $\dfrac{a}{2\sin\frac{\pi}{n}}\sqrt{1+\cos^2\frac{\pi}{n}\tan^2\gamma}$, respectively. 12. The tetrahedron is made coincident with itself by 24 different motions, the cube and octahedron by 48, and the dodecahedron and icosahedron by 120.

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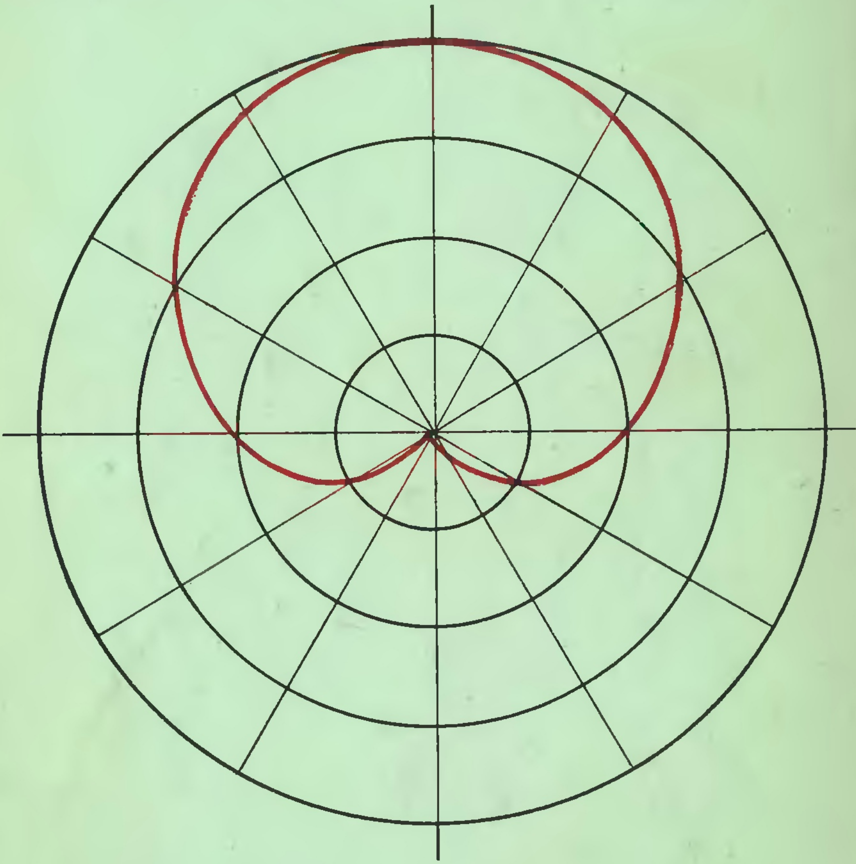

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